8.4. Restricting reloid to a filter. Domain and image

DEFINITION 1016. *Identity reloid* for a set A is defined by the formula $1^{\mathsf{RLD}}_A = \uparrow^{\mathsf{RLD}(A,A)} \mathrm{id}_A$.

OBVIOUS 1017. $(1^{\mathsf{RLD}}_A)^{-1} = 1^{\mathsf{RLD}}_A$.

DEFINITION 1018. I define *restricting* a reloid f to a filter $\mathcal{A}$ as $f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \top^{\mathscr{F}(\mathrm{Dst}\, f)})$.

DEFINITION 1019. *Domain* and *image* of a reloid f are defined as follows:

$$\mathrm{dom}\, f = \bigsqcap^{\mathscr{F}} \langle \mathrm{dom} \rangle^* \,\mathrm{up}\, f; \quad \mathrm{im}\, f = \bigsqcap^{\mathscr{F}} \langle \mathrm{im} \rangle^* \,\mathrm{up}\, f.$$

PROPOSITION 1020. $f \sqsubseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \Leftrightarrow \mathrm{dom}\, f \sqsubseteq \mathcal{A} \wedge \mathrm{im}\, f \sqsubseteq \mathcal{B}$ for every reloid f and filters $\mathcal{A} \in \mathscr{F}(\mathrm{Src}\, f)$, $\mathcal{B} \in \mathscr{F}(\mathrm{Dst}\, f)$.

PROOF.

$\Rightarrow$. It follows from $\mathrm{dom}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \sqsubseteq \mathcal{A} \wedge \mathrm{im}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \sqsubseteq \mathcal{B}$.

$\Leftarrow$. $\mathrm{dom}\, f \sqsubseteq \mathcal{A} \Leftrightarrow \forall A \in \mathrm{up}\, \mathcal{A} \exists F \in \mathrm{up}\, f : \mathrm{dom}\, F \sqsubseteq A$. Analogously $\mathrm{im}\, f \sqsubseteq \mathcal{B} \Leftrightarrow \forall B \in \mathrm{up}\, \mathcal{B} \exists G \in \mathrm{up}\, f : \mathrm{im}\, G \sqsubseteq B$.

Let $\mathrm{dom}\, f \sqsubseteq \mathcal{A} \wedge \mathrm{im}\, f \sqsubseteq \mathcal{B}$, $A \in \mathrm{up}\, \mathcal{A}$, $B \in \mathrm{up}\, \mathcal{B}$. Then there exist $F, G \in \mathrm{up}\, f$ such that $\mathrm{dom}\, F \sqsubseteq A \wedge \mathrm{im}\, G \sqsubseteq B$. Consequently $F \sqcap G \in \mathrm{up}\, f$, $\mathrm{dom}(F \sqcap G) \sqsubseteq A$, $\mathrm{im}(F \sqcap G) \sqsubseteq B$ that is $F \sqcap G \sqsubseteq A \times B$. So there exists $H \in \mathrm{up}\, f$ such that $H \sqsubseteq A \times B$ for every $A \in \mathrm{up}\, \mathcal{A}$, $B \in \mathrm{up}\, \mathcal{B}$. So $f \sqsubseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.

□

DEFINITION 1021. I call *restricted identity reloid* for a filter $\mathcal{A}$ the reloid

$$\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = (1^{\mathsf{RLD}}_{\mathrm{Base}(\mathcal{A})})|_{\mathcal{A}}.$$

THEOREM 1022. $\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = \bigsqcap^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))}_{A \in \mathrm{up}\, \mathcal{A}} \mathrm{id}_A$ for every filter $\mathcal{A}$.

PROOF. Let $K \in \mathrm{up} \bigsqcap^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))}_{A \in \mathrm{up}\, \mathcal{A}} \mathrm{id}_A$, then there exists $A \in \mathrm{up}\, \mathcal{A}$ such that $\mathrm{GR}\, K \supseteq \mathrm{id}_A$. Then

$$\begin{aligned} &\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} \sqsubseteq \\ &\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))} \mathrm{id}_{\mathrm{Base}(\mathcal{A})} \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \top) \sqsubseteq \\ &\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))} \mathrm{id}_{\mathrm{Base}(\mathcal{A})} \sqcap (A \times^{\mathsf{RLD}} \top) = \\ &\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))} \mathrm{id}_{\mathrm{Base}(\mathcal{A})} \sqcap \uparrow^{\mathsf{RLD}} (A \times \top) = \\ &\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))} (\mathrm{id}_{\mathrm{Base}(\mathcal{A})} \cap \mathrm{GR}(A \times \top)) = \\ &\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))} \mathrm{id}_A \sqsubseteq K. \end{aligned}$$

Thus $K \in \mathrm{up}\, \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$.

Reversely let $K \in \mathrm{up}\, \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = \mathrm{up}(1^{\mathsf{RLD}}_{\mathrm{Base}(\mathcal{A})} \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \top))$, then there exists $A \in \mathrm{up}\, \mathcal{A}$ such that

$$\begin{aligned} K \in \mathrm{up} \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))} (\mathrm{id}_{\mathrm{Base}(\mathcal{A})} \cap \mathrm{GR}(A \times \top)) &= \\ \mathrm{up} \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))} \mathrm{id}_A &\supseteq \\ \mathrm{up} \bigsqcap^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))}_{A \in \mathrm{up}\, \mathcal{A}} \mathrm{id}_A&. \end{aligned}$$

□