PROOF.

$$
\begin{aligned}
g \circ f \not\asymp h &\Leftrightarrow \\
\prod^{\mathsf{RLD}} \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\} \sqcap \prod^{\mathsf{RLD}} \operatorname{up} h \neq \bot &\Leftrightarrow \\
\prod^{\mathsf{RLD}} \left\{ \frac{(G \circ F) \sqcap^{\mathsf{RLD}} H}{F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h} \right\} \neq \bot &\Leftrightarrow \\
\prod^{\mathsf{RLD}} \left\{ \frac{(G \circ F) \sqcap H}{F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h} \right\} \neq \bot &\Leftrightarrow \\
\forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : \uparrow^{\mathsf{RLD}} ((G \circ F) \sqcap H) \neq \bot &\Leftrightarrow \\
\forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : G \circ F \not\asymp H &
\end{aligned}
$$

(used properties of generalized filter bases).

Similarly $g \not\asymp h \circ f^{-1} \Leftrightarrow \forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : G \not\asymp H \circ F^{-1}$.

Thus $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}$ because $G \circ F \not\asymp H \Leftrightarrow G \not\asymp H \circ F^{-1}$ by proposition 280. □

THEOREM 1007. For every composable reloids f and g

1°. $g \circ f = \bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\}$.

2°. $g \circ f = \bigsqcup \left\{ \frac{G \circ f}{G \in \operatorname{atoms} g} \right\}$.

PROOF. We will prove only the first as the second is dual. □

Obviously $\bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\} \sqsubseteq g \circ f$. We need to prove $\bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\} \sqsupseteq g \circ f$. Really,

$$
\begin{aligned}
\bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\} \sqsupseteq g \circ f &\Leftrightarrow \\
\forall x \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} g) : \left(x \not\asymp g \circ f \Rightarrow x \not\asymp \bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\} \right) &\Leftarrow \\
\forall x \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} g) : (x \not\asymp g \circ f \Rightarrow \exists F \in \operatorname{atoms} f : x \not\asymp g \circ F) &\Leftrightarrow \\
\forall x \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} g) : (g^{-1} \circ x \not\asymp f \Rightarrow \exists F \in \operatorname{atoms} f : g^{-1} \circ x \not\asymp F) &
\end{aligned}
$$

what is obviously true.

COROLLARY 1008. If f and g are composable reloids, then

$$g \circ f = \bigsqcup \left\{ \frac{G \circ F}{F \in \operatorname{atoms} f, G \in \operatorname{atoms} g} \right\}.$$

PROOF. $g \circ f = \bigsqcup_{F \in \operatorname{atoms} f} (g \circ F) = \bigsqcup_{F \in \operatorname{atoms} f} \bigsqcup_{G \in \operatorname{atoms} g} (G \circ F) = \bigsqcup \left\{ \frac{G \circ F}{F \in \operatorname{atoms} f, G \in \operatorname{atoms} g} \right\}$. □

8.3. Reloidal product of filters

DEFINITION 1009. Reloidal product of filters $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \overset{\text{def}}{=} \prod^{\mathsf{RLD}} \left\{ \frac{A \times B}{A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}} \right\}.$$

OBVIOUS 1010.

- $\uparrow^U A \times^{\mathsf{RLD}} \uparrow^V B = \uparrow^{\mathsf{RLD}(U,V)} (A \times B)$ for every sets $A \subseteq U$, $B \subseteq V$.
- $\uparrow A \times^{\mathsf{RLD}} \uparrow B = \uparrow^{\mathsf{RLD}} (A \times B)$ for every typed sets A, B.