OBVIOUS 996. $(\mathsf{RLD}(A,B),\mathbf{Rel}(A,B))$ is a powerset filtrator isomorphic to the filtrator $(\mathsf{RLD}\sharp(A,B),\mathbf{Rel}(A,B))$. Thus $\mathsf{RLD}(A,B)$ is a special case of $\mathsf{RLD}\sharp(A,B)$.

8.2. Composition of reloids

DEFINITION 997. Reloids f and g are *composable* when $\operatorname{Dst} f = \operatorname{Src} g$.

DEFINITION 998. *Composition* of (composable) reloids is defined by the formula

$$g \circ f = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\}.$$

OBVIOUS 999. Composition of reloids is a reloid.

OBVIOUS 1000. $\uparrow^{\mathsf{RLD}} g \circ \uparrow^{\mathsf{RLD}} f = \uparrow^{\mathsf{RLD}} (g \circ f)$ for composable morphisms f, g of category **Rel**.

THEOREM 1001. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable reloids f, g, h.

PROOF. For two nonempty collections A and B of sets I will denote

$$A \sim B \Leftrightarrow \forall K \in A \exists L \in B : L \subseteq K \wedge \forall K \in B \exists L \in A : L \subseteq K.$$

It is easy to see that $\sim$ is a transitive relation.

I will denote $B \circ A = \left\{ \frac{L \circ K}{K \in A, L \in B} \right\}$.

Let first prove that for every nonempty collections of relations A, B, C

$$A \sim B \Rightarrow A \circ C \sim B \circ C.$$

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B : K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for every $P \in A \circ C$ there exists $P' \in B \circ C$ such that $P' \subseteq P$; the vice versa is analogous. So $A \circ C \sim B \circ C$.

$\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ g) \circ \operatorname{up} f$, $\operatorname{up}(h \circ g) \sim (\operatorname{up} h) \circ (\operatorname{up} g)$. By proven above $\operatorname{up}((h \circ g) \circ f) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

Analogously $\operatorname{up}(h \circ (g \circ f)) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

So $\operatorname{up}(h \circ (g \circ f)) \sim \operatorname{up}((h \circ g) \circ f)$ what is possible only if $\operatorname{up}(h \circ (g \circ f)) = \operatorname{up}((h \circ g) \circ f)$. Thus $(h \circ g) \circ f = h \circ (g \circ f)$. □

EXERCISE 1002. Prove $f_n \circ \cdots \circ f_0 = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F_n \circ \cdots \circ F_0}{F_i \in \operatorname{up} f_i} \right\}$ for every composable reloids $f_0, \ldots, f_n$ where n is an integer, independently of the inserted parentheses. (Hint: Use generalized filter bases.)

THEOREM 1003. For every reloid f:

1°. $f \circ f = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F \circ F}{F \in \operatorname{up} f} \right\}$ if $\operatorname{Src} f = \operatorname{Dst} f$;

2°. $f^{-1} \circ f = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F^{-1} \circ F}{F \in \operatorname{up} f} \right\}$;

3°. $f \circ f^{-1} = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F \circ F^{-1}}{F \in \operatorname{up} f} \right\}$.

PROOF. I will prove only 1° and 2° because 3° is analogous to 2°.

1°. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f : H \circ H \sqsubseteq G \circ F$. To prove it take $H = F \sqcap G$.

2°. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f : H^{-1} \circ H \sqsubseteq G^{-1} \circ F$. To prove it take $H = F \sqcap G$. Then $H^{-1} \circ H = (F \sqcap G)^{-1} \circ (F \sqcap G) \sqsubseteq G^{-1} \circ F$.

□

EXERCISE 1004. Prove $f^n = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F^n}{F \in \operatorname{up} f} \right\}$ for every endofuncoid f and positive integer n.