7.13. Funcoids corresponding to pretopologies

Let Δ be a pretopology on a set U and cl the preclosure corresponding to it (see theorem 774).

Both induce a funcoid, I will show that these two funcoids are reverse of each other:

THEOREM 931. Let f be a complete funcoid defined by the formula $\langle f\rangle^* @\{x\} = \Delta(x)$ for every $x \in U$, let g be a co-complete funcoid defined by the formula $\langle g\rangle^* X = \uparrow^U \operatorname{cl}(\operatorname{GR} X)$ for every $X \in \mathscr{T} U$. Then $g = f^{-1}$.

REMARK 932. It is obvious that funcoids f and g exist.

PROOF. For $X, Y \in \mathscr{T} U$ we have

$$X\ [g]^*\ Y \Leftrightarrow$$
$$\uparrow Y \not\asymp \langle g\rangle \uparrow X \Leftrightarrow$$
$$Y \not\asymp \operatorname{cl}(\operatorname{GR} X) \Leftrightarrow$$
$$\exists y \in Y : \Delta(y) \not\asymp \uparrow X \Leftrightarrow$$
$$\exists y \in Y : \langle f\rangle^* \uparrow^U \{y\} \not\asymp \uparrow X \Leftrightarrow$$

(proposition 607 and properties of complete funcoids)

$$\langle f\rangle^* Y \not\asymp \uparrow X \Leftrightarrow$$
$$Y\ [f]^*\ X.$$

So $g = f^{-1}$. □

7.14. Completion of funcoids

THEOREM 933. $\operatorname{Cor} f = \operatorname{Cor}' f$ for an element f of a filtrator of funcoids.

PROOF. By theorem 542 and corollary 922. □

DEFINITION 934. *Completion* of a funcoid $f \in \mathsf{FCD}(A, B)$ is the complete funcoid $\operatorname{Compl} f \in \mathsf{FCD}(A, B)$ defined by the formula $\langle \operatorname{Compl} f\rangle^* @\{\alpha\} = \langle f\rangle^* @\{\alpha\}$ for $\alpha \in \operatorname{Src} f$.

DEFINITION 935. *Co-completion* of a funcoid f is defined by the formula

$$\operatorname{CoCompl} f = (\operatorname{Compl} f^{-1})^{-1}.$$

OBVIOUS 936. $\operatorname{Compl} f \sqsubseteq f$ and $\operatorname{CoCompl} f \sqsubseteq f$.

PROPOSITION 937. The filtrator $(\mathsf{FCD}(A, B), \mathsf{ComplFCD}(A, B))$ is filtered.

PROOF. Because the filtrator of funcoids is filtered. □

THEOREM 938. $\operatorname{Compl} f = \operatorname{Cor}^{\mathsf{ComplFCD}(A,B)} f = \operatorname{Cor}'^{\mathsf{ComplFCD}(A,B)} f$ for every funcoid $f \in \mathsf{FCD}(A, B)$.

PROOF. $\operatorname{Cor}^{\mathsf{ComplFCD}(A,B)} f = \operatorname{Cor}'^{\mathsf{ComplFCD}(A,B)} f$ using theorem 542 since the filtrator $(\mathsf{FCD}(A, B), \mathsf{ComplFCD}(A, B))$ is filtered.

Let $g \in \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f$. Then $g \in \mathsf{ComplFCD}(A, B)$ and $g \sqsupseteq f$. Thus $g = \operatorname{Compl} g \sqsupseteq \operatorname{Compl} f$.

Thus $\forall g \in \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f : g \sqsupseteq \operatorname{Compl} f$.

Let $\forall g \in \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f : h \sqsubseteq g$ for some $h \in \mathsf{ComplFCD}(A, B)$.

Then $h \sqsubseteq \bigsqcap \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f = f$ and consequently $h = \operatorname{Compl} h \sqsubseteq \operatorname{Compl} f$.

Thus

$$\operatorname{Compl} f = \bigsqcap^{\mathsf{ComplFCD}(A,B)} \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f = \operatorname{Cor}^{\mathsf{ComplFCD}(A,B)} f.$$