Proof. First, it's easy to see that $\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ are elements of $\mathsf{ComplFCD}(A,B)$. Also $\bot^{\mathsf{FCD}(A,B)}$ is an element of $\mathsf{ComplFCD}(A,B)$.

$\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ are atoms of $\mathsf{ComplFCD}(A,B)$ because they are atoms of $\mathsf{FCD}(A,B)$.

It remains to prove that if f is an atom of $\mathsf{ComplFCD}(A,B)$ then $f =\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ for some $\alpha \in A$ and an ultrafilter b on B.

Suppose $f \in \mathsf{FCD}(A,B)$ is a non-empty complete funcoid. Then there exists $\alpha \in A$ such that $\langle f\rangle^* @\{\alpha\} \neq \bot^{\mathscr{F}(B)}$. Thus $\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b \sqsubseteq f$ for some ultrafilter b on B. If f is an atom then $f =\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$. □

Theorem 928. $G \mapsto \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ is an order isomorphism from the set of functions $G \in \mathscr{F}(B)^A$ to the set $\mathsf{ComplFCD}(A,B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \langle f\rangle^* @\{\alpha\}$ where f is a complete funcoid.

Proof. $\bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ is complete because $G(\alpha) = \bigsqcup \operatorname{atoms} G(\alpha)$ and thus

$$\bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha)) = \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b}{\alpha \in A, b \in \operatorname{atoms} G(\alpha)}\right\}$$

is complete. So $G \mapsto \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ is a function from $G \in \mathscr{F}(B)^A$ to $\mathsf{ComplFCD}(A,B)$.

Let f be complete. Then take

$$G(\alpha) = \bigsqcup\left\{\frac{b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}}{\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b \sqsubseteq f}\right\}$$

and we have $f = \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ obviously. So $G \mapsto \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ is surjection onto $\mathsf{ComplFCD}(A,B)$.

Let now prove that it is an injection:

Let

$$f = \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} F(\alpha)) = \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$$

for some $F, G \in \mathscr{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$. We need to prove $F = G$. Let $\beta \in \operatorname{Src} f$.

$$\langle f\rangle^* @\{\beta\} = \bigsqcup_{\alpha\in A}\left\langle\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} F(\alpha)\right\rangle^* @\{\beta\} = F(\beta).$$

Similarly $\langle f\rangle^* @\{\beta\} = G(\beta)$. So $F(\beta) = G(\beta)$.

We have proved that it is a bijection. To show that it is monotone is trivial.

Denote $f = \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$. Then

$$\langle f\rangle^* @\{\alpha'\} = (\text{because } \uparrow^A \{\alpha'\} \text{ is principal}) =$$
$$\bigsqcup_{\alpha\in A}\left\langle\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha)\right\rangle @\{\alpha'\} = \left\langle\uparrow^A \{\alpha'\} \times^{\mathsf{FCD}} G(\alpha')\right\rangle @\{\alpha'\} = G(\alpha').$$

□

Corollary 929. $G \mapsto \bigsqcup_{\alpha\in A}(G(\alpha) \times^{\mathsf{FCD}} \uparrow^A \{\alpha\})$ is an order isomorphism from the set of functions $G \in \mathscr{F}(B)^A$ to the set $\mathsf{CoComplFCD}(A,B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \left\langle f^{-1}\right\rangle^* @\{\alpha\}$ where f is a co-complete funcoid.

Corollary 930. $\mathsf{ComplFCD}(A,B)$ and $\mathsf{CoComplFCD}(A,B)$ are co-frames.