2°⇒3°, 4° ⇒5°, 5°⇒3°, 5°⇒6°. Obvious.

□

The following proposition shows that complete funcoids are a direct generalization of pretopological spaces.

PROPOSITION 917. To specify a complete funcoid f it is enough to specify $\langle f\rangle^*$ on one-element sets, values of $\langle f\rangle^*$ on one element sets can be specified arbitrarily.

PROOF. From the above theorem is clear that knowing $\langle f\rangle^*$ on one-element sets $\langle f\rangle^*$ can be found on every set and then the value of $\langle f\rangle$ can be inferred for every filter.

Choosing arbitrarily the values of $\langle f\rangle^*$ on one-element sets we can define a complete funcoid the following way: $\langle f\rangle^* X = \bigsqcup_{\alpha\in\operatorname{atoms} X}\langle f\rangle^*\alpha$ for every $X \in \mathscr{T}(\operatorname{Src} f)$. Obviously it is really a complete funcoid. □

THEOREM 918. A funcoid is principal iff it is both complete and co-complete.

PROOF.

⇒. Obvious.

⇐. Let f be both a complete and co-complete funcoid. Consider the relation g defined by that $\uparrow\langle g\rangle^*\alpha = \langle f\rangle^*\alpha$ for one-element sets α (g is correctly defined because f corresponds to a generalized closure). Because f is a complete funcoid f is the funcoid corresponding to g.

□

THEOREM 919. If $R \in \mathscr{P}\mathsf{FCD}(A, B)$ is a set of (co-)complete funcoids then $\bigsqcup R$ is a (co-)complete funcoid (for every sets A and B).

PROOF. It is enough to prove for co-complete funcoids. Let $R \in \mathscr{P}\mathsf{FCD}(A, B)$ be a set of co-complete funcoids. Then for every $X \in \mathscr{T}(\operatorname{Src} f)$

$$\left\langle\bigsqcup R\right\rangle^* X = \bigsqcup_{f\in R}\langle f\rangle^* X$$

is a principal filter (used theorem 849). □

COROLLARY 920. If R is a set of binary relations between sets A and B then $\bigsqcup\left\langle\uparrow^{\mathsf{FCD}(A,B)}\right\rangle^* R = \uparrow^{\mathsf{FCD}(A,B)}\bigcup R$.

PROOF. From two last theorems. □

LEMMA 921. Every funcoid is representable as meet (on the lattice of funcoids) of binary relations of the form $X \times Y \sqcup \overline{X} \times \top^{\mathscr{T}(B)}$ (where X, Y are typed sets).

PROOF. Let $f \in \mathsf{FCD}(A, B)$, $X \in \mathscr{T}A$, $Y \in \operatorname{up}\langle f\rangle X$, $g(X, Y) \stackrel{\text{def}}{=} X \times Y \sqcup \overline{X} \times \top^{\mathscr{T}(B)}$. Then $g(X, Y) = X \times^{\mathsf{FCD}} Y \sqcup \overline{X} \times^{\mathsf{FCD}} \top^{\mathscr{F}(B)}$. For every $K \in \mathscr{T}A$

$$\langle g(X,Y)\rangle^* K = \left\langle X \times^{\mathsf{FCD}} Y\right\rangle^* K \sqcup \left\langle\overline{X} \times^{\mathsf{FCD}} \top^{\mathscr{F}(B)}\right\rangle^* K = \left(\begin{cases} \bot^{\mathscr{F}(B)} & \text{if } K = \bot^{\mathscr{T}A} \\ Y & \text{if } \bot^{\mathscr{T}A} \neq K \sqsubseteq X \\ \top^{\mathscr{F}(B)} & \text{if } K \not\sqsubseteq X \end{cases}\right) \sqsupseteq \langle f\rangle^* K;$$

so $g(X, Y) \sqsupseteq f$. For every $X \in \mathscr{T}A$

$$\bigsqcap_{Y\in\operatorname{up}\langle f\rangle^* X}\langle g(X,Y)\rangle^* X = \bigsqcap^{\mathscr{F}}_{Y\in\operatorname{up}\langle f\rangle^* X} Y = \langle f\rangle^* X;$$