PROOF.

$$\begin{aligned} g\circ f\not\asymp h\Leftrightarrow\\ \exists a\in\operatorname{atoms}^{\mathscr{F}(A)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:a\;[(g\circ f)\sqcap h]\;c\Leftrightarrow\\ \exists a\in\operatorname{atoms}^{\mathscr{F}(A)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:(a\;[g\circ f]\;c\wedge a\;[h]\;c)\Leftrightarrow\\ \exists a\in\operatorname{atoms}^{\mathscr{F}(A)},b\in\operatorname{atoms}^{\mathscr{F}(B)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:(a\;[f]\;b\wedge b\;[g]\;c\wedge a\;[h]\;c)\Leftrightarrow\\ \exists b\in\operatorname{atoms}^{\mathscr{F}(B)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:(b\;[g]\;c\wedge b\;\left[h\circ f^{-1}\right]\;c)\Leftrightarrow\\ \exists b\in\operatorname{atoms}^{\mathscr{F}(B)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:b\;\left[g\sqcap(h\circ f^{-1})\right]\;c\Leftrightarrow\\ g\not\asymp h\circ f^{-1}. \end{aligned}$$

□

7.10. Funcoidal product of filters

A generalization of Cartesian product of two sets is funcoidal product of two filters:

DEFINITION 880. *Funcoidal product* of filters $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\in\mathsf{FCD}(\operatorname{Base}(\mathcal{A}),\operatorname{Base}(\mathcal{B}))$ that for every $\mathcal{X}\in\operatorname{Base}(\mathcal{A})$, $\mathcal{Y}\in\operatorname{Base}(\mathcal{B})$

$$\mathcal{X}\;\left[\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\right]\;\mathcal{Y}\Leftrightarrow\mathcal{X}\not\asymp\mathcal{A}\wedge\mathcal{Y}\not\asymp\mathcal{B}.$$

PROPOSITION 881. $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ is really a funcoid and

$$\left\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\right\rangle\mathcal{X}=\begin{cases}\mathcal{B} & \text{if }\mathcal{X}\not\asymp\mathcal{A}\\ \bot^{\mathscr{F}(\operatorname{Base}(\mathcal{B}))} & \text{if }\mathcal{X}\asymp\mathcal{A}.\end{cases}$$

PROOF. Obvious. □

OBVIOUS 882.

- $\uparrow^{\mathsf{FCD}(U,V)}(A\times B)=\uparrow^U A\times\uparrow^V B$ for sets $A\subseteq U$ and $B\subseteq V$.
- $\uparrow^{\mathsf{FCD}}(A\times B)=\uparrow A\times\uparrow B$ for typed sets A and B.

PROPOSITION 883. $f\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\Leftrightarrow\operatorname{dom}f\sqsubseteq\mathcal{A}\wedge\operatorname{im}f\sqsubseteq\mathcal{B}$ for every $f\in\mathsf{FCD}(A,B)$ and $\mathcal{A}\in\mathscr{F}(A)$, $\mathcal{B}\in\mathscr{F}(B)$.

PROOF. If $f\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ then $\operatorname{dom}f\sqsubseteq\operatorname{dom}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\sqsubseteq\mathcal{A}$, $\operatorname{im}f\sqsubseteq\operatorname{im}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\sqsubseteq\mathcal{B}$. If $\operatorname{dom}f\sqsubseteq\mathcal{A}\wedge\operatorname{im}f\sqsubseteq\mathcal{B}$ then

$$\forall\mathcal{X}\in\mathscr{F}(A),\mathcal{Y}\in\mathscr{F}(B):(\mathcal{X}\;[f]\;\mathcal{Y}\Rightarrow\mathcal{X}\sqcap\mathcal{A}\neq\bot\wedge\mathcal{Y}\sqcap\mathcal{B}\neq\bot);$$

consequently $f\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of a meet on the lattice of funcoids:

THEOREM 884. $f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})=\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\circ f\circ\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$ for every funcoid f and $\mathcal{A}\in\mathscr{F}(\operatorname{Src}f)$, $\mathcal{B}\in\mathscr{F}(\operatorname{Dst}f)$.

PROOF. $h\stackrel{\text{def}}{=}\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\circ f\circ\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$. For every $\mathcal{X}\in\mathscr{F}(\operatorname{Src}f)$

$$\langle h\rangle\mathcal{X}=\left\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\right\rangle\langle f\rangle\left\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}\right\rangle\mathcal{X}=\mathcal{B}\sqcap\langle f\rangle(\mathcal{A}\sqcap\mathcal{X}).$$

From this, as easy to show, $h\sqsubseteq f$ and $h\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. If $g\sqsubseteq f\wedge g\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ for a $g\in\mathsf{FCD}(\operatorname{Src}f,\operatorname{Dst}f)$ then $\operatorname{dom}g\sqsubseteq\mathcal{A}$, $\operatorname{im}g\sqsubseteq\mathcal{B}$,

$$\langle g\rangle\mathcal{X}=\mathcal{B}\sqcap\langle g\rangle(\mathcal{A}\sqcap\mathcal{X})\sqsubseteq\mathcal{B}\sqcap\langle f\rangle(\mathcal{A}\sqcap\mathcal{X})=\left\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\right\rangle\langle f\rangle\left\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}\right\rangle\mathcal{X}=\langle h\rangle\mathcal{X},$$

$g\sqsubseteq h$. So $h=f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})$. □