The reverse of (12) implication is trivial, so

$$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow X, y \in \operatorname{atoms} \uparrow Y : x \ \delta \ y \Leftrightarrow a \ \delta \ b.$$

Also

$$\begin{aligned} &\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow X, y \in \operatorname{atoms} \uparrow Y : x \ \delta \ y \Leftrightarrow \\ &\forall X \in \operatorname{up} a, Y \in \operatorname{up} b : X \ \delta' \ Y \Leftrightarrow \\ &a \ [f] \ b. \end{aligned}$$

So $a \ \delta \ b \Leftrightarrow a \ [f] \ b$, that is $[f]$ is a continuation of δ.

□

One of uses of the previous theorem is the proof of the following theorem:

THEOREM 875. If A and B are sets, $R \in \mathscr{P}\mathsf{FCD}(A, B)$, $x \in \operatorname{atoms}^{\mathscr{F}(A)}$, $y \in \operatorname{atoms}^{\mathscr{F}(B)}$, then

1°. $\langle \bigsqcap R \rangle x = \bigsqcap_{f \in R} \langle f \rangle x$;
2°. $x \ [\bigsqcap R] \ y \Leftrightarrow \forall f \in R : x \ [f] \ y$.

PROOF.

2°. Let denote $x \ \delta \ y \Leftrightarrow \forall f \in R : x \ [f] \ y$. For every $a \in \operatorname{atoms}^{\mathscr{F}(A)}$, $b \in \operatorname{atoms}^{\mathscr{F}(B)}$

$$\begin{aligned} &\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow X, y \in \operatorname{atoms} \uparrow Y : x \ \delta \ y \Rightarrow \\ &\forall f \in R, X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow X, y \in \operatorname{atoms} \uparrow Y : x \ [f] \ y \Rightarrow \\ &\forall f \in R, X \in \operatorname{up} a, Y \in \operatorname{up} b : X \ [f]^* \ Y \Rightarrow \\ &\forall f \in R : a \ [f] \ b \Leftrightarrow \\ &a \ \delta \ b. \end{aligned}$$

So by theorem 874, δ can be continued till $[p]$ for some funcoid $p \in \mathsf{FCD}(A, B)$. For every funcoid $q \in \mathsf{FCD}(A, B)$ such that $\forall f \in R : q \sqsubseteq f$ we have

$$x \ [q] \ y \Rightarrow \forall f \in R : x \ [f] \ y \Leftrightarrow x \ \delta \ y \Leftrightarrow x \ [p] \ y,$$

so $q \sqsubseteq p$. Consequently $p = \bigsqcap R$.
From this $x \ [\bigsqcap R] \ y \Leftrightarrow \forall f \in R : x \ [f] \ y$.
1°. From the former

$$\begin{aligned} &y \in \operatorname{atoms}\left\langle \bigsqcap R \right\rangle x \Leftrightarrow \\ &y \sqcap \left\langle \bigsqcap R \right\rangle x \neq \bot \Leftrightarrow \\ &\forall f \in R : y \sqcap \langle f \rangle x \neq \bot \Leftrightarrow \\ &y \in \bigsqcap \langle \operatorname{atoms} \rangle^* \left\{ \frac{\langle f \rangle x}{f \in R} \right\} \Leftrightarrow \\ &y \in \operatorname{atoms} \bigsqcap_{f \in R} \langle f \rangle x \end{aligned}$$

for every $y \in \operatorname{atoms}^{\mathscr{F}(A)}$. From this it follows $\langle \bigsqcap R \rangle x = \bigsqcap_{f \in R} \langle f \rangle x$.

□

THEOREM 876. $g \circ f = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\}$ for every composable funcoids f and g.