Proof. For every $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} g)$ we have

$$\begin{aligned} \mathcal{X}\ [g \circ f]\ \mathcal{Y} &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle g \circ f \rangle \mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle g \rangle \langle f \rangle \mathcal{X} \neq \bot &\Leftrightarrow \\ \langle f \rangle \mathcal{X}\ [g]\ \mathcal{Y} &\Leftrightarrow \\ \mathcal{X}\ ([g] \circ \langle f \rangle)\ \mathcal{Y} & \end{aligned}$$

and

$$\begin{aligned} [g \circ f] &= \\ \left[(f^{-1} \circ g^{-1})^{-1}\right] &= \\ \left[f^{-1} \circ g^{-1}\right]^{-1} &= \\ (\left[f^{-1}\right] \circ \left\langle g^{-1} \right\rangle)^{-1} &= \\ \left\langle g^{-1} \right\rangle^{-1} \circ [f]. & \end{aligned}$$

□

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x\ (g \circ f)\ z \Leftrightarrow \exists y : (x\ f\ y \wedge y\ g\ z)$ for every x and z and every binary relations f and g.

Theorem 852. For every sets A, B, C and $f \in \mathsf{FCD}(A, B)$, $g \in \mathsf{FCD}(B, C)$ and $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Z} \in \mathscr{F}(C)$

$$\mathcal{X}\ [g \circ f]\ \mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X}\ [f]\ y \wedge y\ [g]\ \mathcal{Z}).$$

Proof.

$$\begin{aligned} \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X}\ [f]\ y \wedge y\ [g]\ \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{Z} \sqcap \langle g \rangle y \neq \bot \wedge y \sqcap \langle f \rangle \mathcal{X} \neq \bot) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{Z} \sqcap \langle g \rangle y \neq \bot \wedge y \sqsubseteq \langle f \rangle \mathcal{X}) &\Rightarrow \\ \mathcal{Z} \sqcap \langle g \rangle \langle f \rangle \mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{X}\ [g \circ f]\ \mathcal{Z}. & \end{aligned}$$

Reversely, if $\mathcal{X}\ [g \circ f]\ \mathcal{Z}$ then $\langle f \rangle \mathcal{X}\ [g]\ \mathcal{Z}$, consequently there exists $y \in \operatorname{atoms} \langle f \rangle \mathcal{X}$ such that $y\ [g]\ \mathcal{Z}$; we have $\mathcal{X}\ [f]\ y$. □

Theorem 853. For every sets A, B, C

1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for $g, h \in \mathsf{FCD}(A, B)$, $f \in \mathsf{FCD}(B, C)$;
2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$ for $g, h \in \mathsf{FCD}(B, C)$, $f \in \mathsf{FCD}(A, B)$.

Proof. I will prove only the first equality because the other is analogous.
For every $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Z} \in \mathscr{F}(C)$

$$\begin{aligned} \mathcal{X}\ [f \circ (g \sqcup h)]\ \mathcal{Z} &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X}\ [g \sqcup h]\ y \wedge y\ [f]\ \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : ((\mathcal{X}\ [g]\ y \vee \mathcal{X}\ [h]\ y) \wedge y\ [f]\ \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : ((\mathcal{X}\ [g]\ y \wedge y\ [f]\ \mathcal{Z}) \vee (\mathcal{X}\ [h]\ y \wedge y\ [f]\ \mathcal{Z})) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X}\ [g]\ y \wedge y\ [f]\ \mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X}\ [h]\ y \wedge y\ [f]\ \mathcal{Z}) &\Leftrightarrow \\ \mathcal{X}\ [f \circ g]\ \mathcal{Z} \vee \mathcal{X}\ [f \circ h]\ \mathcal{Z} &\Leftrightarrow \\ \mathcal{X}\ [f \circ g \sqcup f \circ h]\ \mathcal{Z}. & \end{aligned}$$

□