1°.

$$\begin{aligned} X \left[\bigsqcup R\right]^* Y &\Leftrightarrow \\ \uparrow Y \sqcap \left\langle \bigsqcup R\right\rangle^* X \neq \bot &\Leftrightarrow \\ \uparrow Y \sqcap \bigsqcup_{f\in R} \langle f\rangle^* X \neq \bot &\Leftrightarrow \\ \exists f\in R : \uparrow Y \sqcap \langle f\rangle^* X \neq \bot &\Leftrightarrow \\ \exists f\in R : X\, [f]^*\, Y& \end{aligned}$$

(used proposition 607).

□

In the next theorem, compared to the previous one, the class of infinite joins is replaced with lesser class of binary joins and simultaneously class of sets is changed to more wide class of filters.

THEOREM 850. For every $f, g \in \mathsf{FCD}(A, B)$ and $\mathcal{X} \in \mathscr{F}(A)$ (for every sets A, B)

1°. $\langle f \sqcup g\rangle \mathcal{X} = \langle f\rangle \mathcal{X} \sqcup \langle g\rangle \mathcal{X}$;
2°. $[f \sqcup g] = [f] \cup [g]$.

PROOF.

1°. Let $\alpha\mathcal{X} \overset{\text{def}}{=} \langle f\rangle \mathcal{X} \sqcup \langle g\rangle \mathcal{X}$; $\beta\mathcal{Y} \overset{\text{def}}{=} \left\langle f^{-1}\right\rangle \mathcal{Y} \sqcup \left\langle g^{-1}\right\rangle \mathcal{Y}$ for every $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Y} \in \mathscr{F}(B)$. Then

$$\begin{aligned} \mathcal{Y} \sqcap \alpha\mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle f\rangle \mathcal{X} \neq \bot \vee \mathcal{Y} \sqcap \langle g\rangle \mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{X} \sqcap \left\langle f^{-1}\right\rangle \mathcal{Y} \neq \bot \vee \mathcal{X} \sqcap \left\langle g^{-1}\right\rangle \mathcal{Y} \neq \bot &\Leftrightarrow \\ \mathcal{X} \sqcap \beta\mathcal{Y} \neq \bot&. \end{aligned}$$

So $h = (A, B, \alpha, \beta)$ is a funcoid. Obviously $h \sqsupseteq f$ and $h \sqsupseteq g$. If $p \sqsupseteq f$ and $p \sqsupseteq g$ for some funcoid p then $\langle p\rangle \mathcal{X} \sqsupseteq \langle f\rangle \mathcal{X} \sqcup \langle g\rangle \mathcal{X} = \langle h\rangle \mathcal{X}$ that is $p \sqsupseteq h$. So $f \sqcup g = h$.

2°. For every $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Y} \in \mathscr{F}(B)$ we have

$$\begin{aligned} \mathcal{X}\, [f \sqcup g]\, \mathcal{Y} &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle f \sqcup g\rangle \mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{Y} \sqcap (\langle f\rangle \mathcal{X} \sqcup \langle g\rangle \mathcal{X}) \neq \bot &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle f\rangle \mathcal{X} \neq \bot \vee \mathcal{Y} \sqcap \langle g\rangle \mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{X}\, [f]\, \mathcal{Y} \vee \mathcal{X}\, [g]\, \mathcal{Y}&. \end{aligned}$$

□

7.6. More on composition of funcoids

PROPOSITION 851. $[g \circ f] = [g] \circ \langle f\rangle = \left\langle g^{-1}\right\rangle^{-1} \circ [f]$ for every composable funcoids f and g.