PROOF.

$$\mathcal{X}\ [f] \bigsqcap S \Leftrightarrow \bigsqcap S \sqcap \langle f\rangle \mathcal{X} \neq \bot \Leftrightarrow \bigsqcap \langle \langle f\rangle \mathcal{X} \sqcap\rangle^* S \neq \bot \Leftrightarrow$$
$$\text{(by properties of generalized filter bases)} \Leftrightarrow$$
$$\exists \mathcal{Y} \in \langle \langle f\rangle \mathcal{X} \sqcap\rangle^* S : \mathcal{Y} \neq \bot \Leftrightarrow \exists \mathcal{Y} \in S : \langle f\rangle \mathcal{X} \sqcap \mathcal{Y} \neq \bot \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X}\ [f]\ \mathcal{Y}.$$

□

DEFINITION 838. A function f between two posets is said to *preserve filtered meets*, when $f \bigsqcap S = \bigsqcap \langle f\rangle^* S$ whenever $\bigsqcap S$ is defined for a filter base S on the first of the two posets.

THEOREM 839. (discovered by TODD TRIMBLE) A function $\varphi : \mathscr{F}(A) \to \mathscr{F}(B)$ preserves finite joins (including nullary joins) and filtered meets iff there exists a funcoid f such that $\langle f\rangle = \varphi$.

PROOF. Backward implication follows from above.

Let $\psi = \varphi|_{\mathscr{T} A}$. Then ψ preserves bottom element and binary joins. Thus there exists a funcoid f such that $\langle f\rangle^* = \psi$.

It remains to prove that $\langle f\rangle = \varphi$.

Really, $\langle f\rangle \mathcal{X} = \bigsqcap \left\langle \langle f\rangle^* \right\rangle^* \operatorname{up} \mathcal{X} = \bigsqcap \langle \psi\rangle^* \operatorname{up} \mathcal{X} = \bigsqcap \langle \varphi\rangle^* \operatorname{up} \mathcal{X} = \varphi \bigsqcap \operatorname{up} \mathcal{X} = \varphi \mathcal{X}$ for every $\mathcal{X} \in \mathscr{F}(A)$. □

COROLLARY 840. Funcoids f from A to B bijectively correspond by the formula $\langle f\rangle = \varphi$ to functions $\varphi : \mathscr{F}(A) \to \mathscr{F}(B)$ preserving finite joins and filtered meets.

7.4. Another way to represent funcoids as binary relations

This is based on a TODD TRIMBLE's idea.

DEFINITION 841. The binary relation $\xi^{\circledast} \in \mathscr{P}(\mathscr{F}(\operatorname{Src} \xi) \times \mathscr{F}(\operatorname{Dst} \xi))$ for a funcoid ξ is defined by the formula $\mathcal{A}\ \xi^{\circledast}\ \mathcal{B} \Leftrightarrow \mathcal{B} \sqsupseteq \langle \xi\rangle \mathcal{A}$.

DEFINITION 842. The binary relation $\xi^* \in \mathscr{P}(\mathscr{T} \operatorname{Src} \xi \times \mathscr{T} \operatorname{Dst} \xi)$ for a funcoid ξ is defined by the formula

$$A\ \xi^*\ B \Leftrightarrow B \sqsupseteq \langle \xi\rangle A \Leftrightarrow B \in \operatorname{up} \langle \xi\rangle A.$$

PROPOSITION 843. Funcoid ξ can be restored from

1°. the value of $\xi^{\circledast}$;
2°. the value of ξ^*.

PROOF.

1°. The value of $\langle \xi\rangle$ can be restored from $\xi^{\circledast}$.
2°. The value of $\langle \xi\rangle^*$ can be restored from ξ^*.

□

THEOREM 844. Let ν and ξ be composable funcoids. Then:

1°. $\xi^{\circledast} \circ \nu^{\circledast} = (\xi \circ \nu)^{\circledast}$;
2°. $\xi^* \circ \nu^* = (\xi \circ \nu)^*$.

PROOF.

1°.

$$\mathcal{A}\ (\xi^{\circledast} \circ \nu^{\circledast})\ \mathcal{C} \Leftrightarrow \exists \mathcal{B} : \left(\mathcal{A}\ \nu^{\circledast}\ \mathcal{B} \wedge \mathcal{B}\ \xi^{\circledast}\ \mathcal{C}\right) \Leftrightarrow$$
$$\exists \mathcal{B} \in \mathscr{F}(\operatorname{Dst} \nu) : (\mathcal{B} \sqsupseteq \langle \nu\rangle \mathcal{A} \wedge \mathcal{C} \sqsupseteq \langle \xi\rangle \mathcal{B}) \Leftrightarrow$$
$$\mathcal{C} \sqsupseteq \langle \xi\rangle \langle \nu\rangle \mathcal{A} \Leftrightarrow \mathcal{C} \sqsupseteq \langle \xi \circ \nu\rangle \mathcal{A} \Leftrightarrow \mathcal{A}\ (\xi \circ \nu)^{\circledast}\ \mathcal{C}.$$