PROPOSITION 798. Every closure generated by a proximity is a Kuratowski closure.

PROOF. First prove it is a preclosure. $\operatorname{cl}(\emptyset) = \emptyset$ is obvious. $\operatorname{cl}(A) \supseteq A$ is obvious.

$$\operatorname{cl}(A \cup B) = \left\{ \frac{a \in U}{\{a\} \ \delta \ A \cup B} \right\} = \left\{ \frac{a \in U}{\{a\} \ \delta \ A \vee \{a\} \ \delta \ B} \right\} = \left\{ \frac{a \in U}{\{a\} \ \delta \ A} \right\} \cup \left\{ \frac{a \in U}{\{a\} \ \delta \ B} \right\} = \operatorname{cl}(A) \cup \operatorname{cl}(B).$$

It is remained to prove that cl is idempotent, that is $\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)$. It is enough to show $\operatorname{cl}(\operatorname{cl}(A)) \subseteq \operatorname{cl}(A)$ that is if $x \notin \operatorname{cl}(A)$ then $x \notin \operatorname{cl}(\operatorname{cl}(A))$.

If $x \notin \operatorname{cl}(A)$ then $\{x\} \ \bar{\delta} \ A$. So there are $P, Q \in \mathscr{P}U$ such that $\{x\} \ \bar{\delta} \ P$, $A \ \bar{\delta} \ Q$, $P \cup Q = U$. Then $U \setminus Q \subseteq P$, so $\{x\} \ \bar{\delta} \ U \setminus Q$ and hence $x \in Q$. Hence $U \setminus \operatorname{cl}(A) \subseteq Q$, and so $\operatorname{cl}(A) \subseteq U \setminus Q \subseteq P$. Consequently $\{x\} \ \bar{\delta} \ \operatorname{cl}(A)$ and hence $x \notin \operatorname{cl}(\operatorname{cl}(A))$. □

6.5. Definition of uniform spaces

Here I will present the traditional definition of uniform spaces. Below in the chapter about reloids I will present a shortened and more algebraic (however a little less elementary) definition of uniform spaces.

DEFINITION 799. *Uniform space* is a pair (U, D) of a set U and filter $D \in \mathfrak{F}(U \times U)$ (called *uniformity* or the set of *entourages*) such that:

1°. If $F \in D$ then $\mathrm{id}_U \subseteq F$.
2°. If $F \in D$ then there exists $G \in D$ such that $G \circ G \subseteq F$.
3°. If $F \in D$ then $F^{-1} \in D$.