If τ is the pretopology induced by topology π, in turn induced by a Kuratowski closure ρ, then $\tau = \rho$.

$$\operatorname{cl}_\tau(A) =$$
$$\bigcap\left\{\frac{X \in \mathscr{P}U}{X \text{ is a closed set in } \pi, X \supseteq A}\right\} =$$
$$\bigcap\left\{\frac{X \in \mathscr{P}U}{\operatorname{cl}_\rho(X) = X, X \supseteq A}\right\} =$$
$$\bigcap\left\{\frac{\operatorname{cl}_\rho(X)}{X \in \mathscr{P}U, \operatorname{cl}_\rho(X) = X, X \supseteq \operatorname{cl}_\rho(A)}\right\} =$$
$$\bigcap\left\{\frac{\operatorname{cl}_\rho(\operatorname{cl}_\rho(X))}{X = A}\right\} =$$
$$\operatorname{cl}_\rho(\operatorname{cl}_\rho(A)) =$$
$$\operatorname{cl}_\rho(A).$$

□

6.3.1.3. *Topology induced by a metric.*

DEFINITION 791. Every metric space induces a topology in this way: A set X is open iff

$$\forall x \in X \exists \epsilon > 0 : B_r(x) \subseteq X.$$

EXERCISE 792. Prove it is really a topology and this topology is the same as the topology, induced by the pretopology, in turn induced by our metric space.

6.4. Proximity spaces

Let (U, d) be metric space. We will define *distance* between sets $A, B \in \mathscr{P}U$ by the formula

$$d(A, B) = \inf\left\{\frac{d(a, b)}{a \in A, b \in B}\right\}.$$

(Here "inf" denotes infimum on the real line.)

DEFINITION 793. Sets $A, B \in \mathscr{P}U$ are *near* (denoted $A\ \delta\ B$) iff $d(A, B) = 0$.

δ defined in this way (for a metric space) is an example of proximity as defined below.

DEFINITION 794. A *proximity space* is a set (U, δ) conforming to the following axioms (for every $A, B, C \in \mathscr{P}U$):

1°. $A \cap B \neq \emptyset \Rightarrow A\ \delta\ B$;
2°. if $A\ \delta\ B$ then $A \neq \emptyset$ and $B \neq \emptyset$;
3°. $A\ \delta\ B \Rightarrow B\ \delta\ A$ (*symmetry*);
4°. $(A \cup B)\ \delta\ C \Leftrightarrow A\ \delta\ C \vee B\ \delta\ C$;
5°. $C\ \delta\ (A \cup B) \Leftrightarrow C\ \delta\ A \vee C\ \delta\ B$;
6°. $A\ \bar{\delta}\ B$ implies existence of $P, Q \in \mathscr{P}U$ with $A\ \bar{\delta}\ P$, $B\ \bar{\delta}\ Q$ and $P \cup Q = U$.

EXERCISE 795. Show that proximity generated by a metric space is really a proximity (conforms to the above axioms).

DEFINITION 796. *Quasi-proximity* is defined as the above but without the symmetry axiom.

DEFINITION 797. Closure is generated by a proximity by the following formula:

$$\operatorname{cl}(A) = \left\{\frac{a \in U}{\{a\}\ \delta\ A}\right\}.$$