It is left to prove that the functions defined by the above formulas are mutually inverse.

Let cl_0 be a preclosure, let Δ be the pretopology induced by cl_0 by the formula (4), let cl_1 be the preclosure induced by Δ by the formula (3). Let's prove $\mathrm{cl}_1 = \mathrm{cl}_0$. Really,

$$\begin{aligned} x \in \mathrm{cl}_1(A) \Leftrightarrow \\ \Delta(x) \not\asymp \uparrow^U A \Leftrightarrow \\ \forall X \in \mathrm{up}\,\Delta(x) : X \cap A \neq \emptyset \Leftrightarrow \\ \forall X \in \mathscr{P}U : (x \notin \mathrm{cl}_0(U \setminus X) \Rightarrow X \cap A \neq \emptyset) \Leftrightarrow \\ \forall X' \in \mathscr{P}U : (x \notin \mathrm{cl}_0(X') \Rightarrow A \setminus X' \neq \emptyset) \Leftrightarrow \\ \forall X' \in \mathscr{P}U : (A \setminus X' = \emptyset \Rightarrow x \in \mathrm{cl}_0(X')) \Leftrightarrow \\ \forall X' \in \mathscr{P}U : (A \subseteq X' \Rightarrow x \in \mathrm{cl}_0(X')) \Leftrightarrow \\ x \in \mathrm{cl}_0(A). \end{aligned}$$

So $\mathrm{cl}_1(A) = \mathrm{cl}_0(A)$.

Let now Δ_0 be a pretopology, let cl be the closure induced by Δ_0 by the formula (3), let Δ_1 be the pretopology induced by cl by the formula (4). Really

$$\begin{aligned} & A \in \mathrm{up}\,\Delta_1(x) \Leftrightarrow \\ & x \notin \mathrm{cl}(U \setminus A) \Leftrightarrow \\ \Delta_0(x) \asymp \uparrow^U (U \setminus A) \Leftrightarrow & \text{ (proposition 548)} \\ & \uparrow^U A \sqsupseteq \Delta_0(x) \Leftrightarrow \\ & A \in \mathrm{up}\,\Delta_0(x). \end{aligned}$$

So $\Delta_1(x) = \Delta_0(x)$.

That these functions are mutually inverse, is now proved. □

6.2.1. Pretopology induced by a metric. Every metric space induces a pretopology by the formula:

$$\Delta(x) = \prod^{\mathscr{F}U} \left\{ \frac{B_r(x)}{r \in \mathbb{R}, r > 0} \right\}.$$

EXERCISE 775. Show that it is a pretopology.

PROPOSITION 776. The preclosure corresponding to this pretopology is the same as the preclosure of the metric space.

PROOF. I denote the preclosure of the metric space as cl_M and the preclosure corresponding to our pretopology as cl_P. We need to show $\mathrm{cl}_P = \mathrm{cl}_M$. Really:

$$\begin{aligned} \mathrm{cl}_P(A) = \\ \left\{ \frac{x \in U}{A \in \partial\Delta(x)} \right\} = \\ \left\{ \frac{x \in U}{\forall \epsilon > 0 : B_\epsilon(x) \not\asymp A} \right\} = \\ \left\{ \frac{y \in U}{\forall \epsilon > 0 \exists a \in A : d(y, a) < \epsilon} \right\} = \\ \mathrm{cl}_M(A) \end{aligned}$$

for every set $A \in \mathscr{P}U$. □