6.1.1. Open and closed sets.

DEFINITION 765. A set A in a metric space is called *open* when $\forall a \in A \exists r > 0 : B_r(a) \subseteq A$.

DEFINITION 766. A set A in a metric space is closed when its complement $U \setminus A$ is open.

EXERCISE 767. Show that: closed intervals on real line are closed sets, open intervals are open sets.

EXERCISE 768. Show that open balls are open and closed balls are closed.

DEFINITION 769. Closure $\operatorname{cl}(A)$ of a set A in a metric space is the set of points y such that

$$\forall \epsilon > 0 \exists a \in A : d(y, a) < \epsilon.$$

PROPOSITION 770. $\operatorname{cl}(A) \supseteq A$.

PROOF. It follows from $d(a, a) = 0 < \epsilon$. □

EXERCISE 771. Prove $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$ for every subsets A and B of a metric space.

6.2. Pretopological spaces

Pretopological space can be defined in two equivalent ways: a *neighborhood system* or a *preclosure operator*. To be more clear I will call *pretopological space* only the first (neighborhood system) and the second call a *preclosure space*.

DEFINITION 772. *Pretopological space* is a set U together with a filter $\Delta(x)$ on U for every $x \in U$, such that $\uparrow^U \{x\} \sqsubseteq \Delta(x)$. Δ is called a *pretopology* on U. Elements of $\operatorname{up} \Delta(x)$ are called *neighborhoods* of point x.

DEFINITION 773. *Preclosure* on a set U is a unary operation cl on $\mathscr{P}U$ such that for every $A, B \in \mathscr{P}U$:

1°. $\operatorname{cl}(\emptyset) = \emptyset$;
2°. $\operatorname{cl}(A) \supseteq A$;
3°. $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$.

I call a preclosure together with a set U as *preclosure space*.

THEOREM 774. Small pretopological spaces and small preclosure spaces bijectively correspond to each other by the formulas:

$$\operatorname{cl}(A) = \left\{ \frac{x \in U}{A \in \partial \Delta(x)} \right\}; \tag{3}$$

$$\operatorname{up} \Delta(x) = \left\{ \frac{A \in \mathscr{P}U}{x \notin \operatorname{cl}(U \setminus A)} \right\}. \tag{4}$$

PROOF. First let's prove that cl defined by formula (3) is really a preclosure.

$\operatorname{cl}(\emptyset) = \emptyset$ is obvious. If $x \in A$ then $A \in \partial \Delta(x)$ and so $\operatorname{cl}(A) \supseteq A$. $\operatorname{cl}(A \cup B) = \left\{ \frac{x \in U}{A \cup B \in \partial \Delta(x)} \right\} = \left\{ \frac{x \in U}{A \in \partial \Delta(x) \vee B \in \partial \Delta(x)} \right\} = \operatorname{cl}(A) \cup \operatorname{cl}(B)$. So, it is really a preclosure.

Next let's prove that Δ defined by formula (4) is a pretopology. That $\operatorname{up} \Delta(x)$ is an upper set is obvious. Let $A, B \in \operatorname{up} \Delta(x)$. Then $x \notin \operatorname{cl}(U \setminus A) \wedge x \notin \operatorname{cl}(U \setminus B)$; $x \notin \operatorname{cl}(U \setminus A) \cup \operatorname{cl}(U \setminus B) = \operatorname{cl}((U \setminus A) \cup (U \setminus B)) = \operatorname{cl}(U \setminus (A \cap B))$; $A \cap B \in \operatorname{up} \Delta(x)$. We have proved that $\Delta(x)$ is a filter object.

Let's prove $\uparrow^U \{x\} \sqsubseteq \Delta(x)$. If $A \in \operatorname{up} \Delta(x)$ then $x \notin \operatorname{cl}(U \setminus A)$ and consequently $x \notin U \setminus A$; $x \in A$; $A \in \operatorname{up} \uparrow^U \{x\}$. So $\uparrow^U \{x\} \sqsubseteq \Delta(x)$ and thus Δ is a pretopology.