Antisymmetry. Suppose $\mathcal{X} \sqsubseteq \mathcal{Y}$ and $\mathcal{Y} \sqsubseteq \mathcal{X}$. Then $\mathscr{S}\mathcal{X} \sqsubseteq \mathscr{S}\mathcal{Y}$ and $\mathscr{S}\mathcal{Y} \sqsubseteq \mathscr{S}\mathcal{X}$. Thus $\mathscr{S}\mathcal{X} = \mathscr{S}\mathcal{Y}$ and so $\mathscr{S}x = \mathscr{S}y$ for some $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Consequently $\mathscr{S}(x \div B) = \mathscr{S}(y \div B)$ for $B = \operatorname{Base}(x) \sqcup \operatorname{Base}(y)$. Thus $x \div B = y \div B$ and so $x \sim y$, thus $\mathcal{X} = \mathcal{Y}$.

□

THEOREM 722. $[x] \sqsubseteq [y] \Leftrightarrow x \sqsubseteq y$ for filters x and y with the same base set.

PROOF.

$\Leftarrow$. Obvious.

$\Rightarrow$. Let $\operatorname{Base}(x) = \operatorname{Base}(y) = B$. Suppose $[x] \sqsubseteq [y]$. Then there exist $x' \sim x$ and $y' \sim y$ such that $C = \operatorname{Base}(x') = \operatorname{Base}(y')$ (for some set C) and $x' \sqsubseteq y'$.

We have by the lemma $x' \div (B \sqcup C) \sqsubseteq y' \div (B \sqcup C)$.

But $x' \div (B \sqcup C) = x \div (B \sqcup C)$ and $y' \div (B \sqcup C) = y \div (B \sqcup C)$. So $x \div (B \sqcup C) \sqsubseteq y \div (B \sqcup C)$ and thus again applying the lemma $x \sqsubseteq y$.

□

PROPOSITION 723. $\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow \mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$ for every unfixed filters $\mathcal{X}$, $\mathcal{Y}$ and set C.

PROOF. Let $\mathcal{X} \sqsubseteq \mathcal{Y}$. Then there are $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\operatorname{Base}(x) = \operatorname{Base}(y)$ and $x \sqsubseteq y$. Then by proved above $x \div C \sqsubseteq y \div C$ what is equivalent to $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$. □

PROPOSITION 724. If $C \in \mathscr{S}\mathcal{X}$ and $C \in \mathscr{S}\mathcal{Y}$ for unfixed filters $\mathcal{X}$ and $\mathcal{Y}$ then $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$.

PROOF.

$\Leftarrow$. Previous proposition.

$\Rightarrow$. Let $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$. We have some $x \in \mathcal{X}$, $y \in \mathcal{Y}$, such that $\operatorname{Base}(x) = \operatorname{Base}(y)$ and $x \div C \sqsubseteq y \div C$. So $\mathscr{S}(x \div C) \sqsubseteq \mathscr{S}(y \div C)$. But $\mathscr{S}(x \div C) \sim x$ and $\mathscr{S}(y \div C) \sim y$. Thus $\mathscr{S}x \sqsubseteq \mathscr{S}y$ that is $x \sqsubseteq y$ and so $\mathcal{X} \sqsubseteq \mathcal{Y}$.

□

5.39.4. Rebase of unfixed filters.

Proposition 716 allows to define:

DEFINITION 725. $\mathcal{A} \div B = a \div B$ for an unfixed filter $\mathcal{A}$ and arbitrary $a \in \mathcal{A}$.

OBVIOUS 726. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \sqsubseteq A$ for every unfixed filter $\mathcal{X}$ and sets A, B.

Proposition 715 allows to define:

DEFINITION 727. $\mathscr{S}\mathcal{A} = \mathscr{S}a$ for every $a \in \mathcal{A}$ for every unfixed filter $\mathcal{A}$.

THEOREM 728. $\mathscr{S}$ is an order-isomorphism from the poset of unfixed filters to the poset of filters on $\mathfrak{Z}$.

PROOF. We already know that $\mathscr{S}$ is an order embedding. It remains to prove that it is a surjection.

Let $\mathcal{Y}$ be a filter on $\mathfrak{Z}$. Take $\mathfrak{Z} \ni X \in \mathcal{Y}$. Then $\langle X \sqcap \rangle^* \mathcal{Y}$ is a filter on X and $\mathscr{S}[\langle X \sqcap \rangle^* \mathcal{Y}] = \mathscr{S}\langle X \sqcap \rangle^* \mathcal{Y} = \mathcal{Y}$. We have proved that it is a surjection. □

OBVIOUS 729. $\mathcal{A} \div B = \langle B \sqcap \rangle^* \mathscr{S}\mathcal{A}$ for every unfixed filter $\mathcal{A}$.

OBVIOUS 730. If $A \in \mathscr{S}\mathcal{A}$ then $\mathcal{A} \div A \in \mathcal{A}$ for every unfixed filter $\mathcal{A}$.

PROPOSITION 731. If $C \in \mathscr{S}\mathcal{X}$ and $C \in \mathscr{S}\mathcal{Y}$ for unfixed filters $\mathcal{X}$ and $\mathcal{Y}$ then $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} = \mathcal{Y}$.