DEFINITION 713. I will call equivalence classes as *unfixed filters*.

REMARK 714. The word "unfixed" is meant to negate "fixed" (having a particular base) filters.

PROPOSITION 715. $\mathcal{A} \sim \mathcal{B}$ iff $\mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$ for every filters $\mathcal{A}$, $\mathcal{B}$ on sets.[3]

PROOF. Let $\mathcal{A} \sim \mathcal{B}$. Then there is a set P such that $P \in \mathcal{A}$, $P \in \mathcal{B}$ and $\mathcal{A} \cap \mathscr{P}P = \mathcal{B} \cap \mathscr{P}P$. So $\mathscr{S}\mathcal{A} = (\mathcal{A} \cap \mathscr{P}P) \cup \left\{ \frac{K \in \mathfrak{Z}}{K \sqsupseteq P} \right\}$. Similarly $\mathscr{S}\mathcal{B} = (\mathcal{B} \cap \mathscr{P}P) \cup \left\{ \frac{K \in \mathfrak{Z}}{K \sqsupseteq P} \right\}$. Combining, we have $\mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$.

Let now $\mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$. Take $K \in \mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$. Then $\mathcal{A} \div K = \mathcal{B} \div K$ and thus (proposition 711) $\mathcal{A} \sim \mathcal{A} \div K = \mathcal{B} \div K \sim \mathcal{B}$, so having $\mathcal{A} \sim \mathcal{B}$. □

PROPOSITION 716. $\mathcal{A} \sim \mathcal{B} \Rightarrow \mathcal{A} \div B = \mathcal{B} \div B$ for every filters $\mathcal{A}$ and $\mathcal{B}$ and set B.

PROOF. $\mathcal{A} \div B = \langle B \sqcap \rangle^* \mathscr{S}\mathcal{A} = \langle B \sqcap \rangle^* \mathscr{S}\mathcal{B} = \mathcal{B} \div B$. □

5.39.3. Poset of unfixed filters.

LEMMA 717. Let filters $\mathcal{X}$ and $\mathcal{Y}$ be such that $\operatorname{Base}(\mathcal{X}) = \operatorname{Base}(\mathcal{Y}) = B$. Then $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$ for every set $C \supseteq B$.

PROOF. $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \div C \supseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} \supseteq \mathcal{Y} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} \Leftrightarrow \mathcal{X} \supseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$. □

PROPOSITION 718. $\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow \mathcal{X} \div B \sqsubseteq \mathcal{Y} \div B$ for every filters $\mathcal{X}$, $\mathcal{Y}$ with the same base and set B.

PROOF. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \supseteq \mathcal{Y} \Rightarrow \mathcal{X} \div B \supseteq \mathcal{Y} \div B \Leftrightarrow \mathcal{X} \div B \sqsubseteq \mathcal{Y} \div B$. □

Define order of unfixed filters using already defined order of filters of a fixed base:

DEFINITION 719. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \exists x \in \mathcal{X}, y \in \mathcal{Y} : (\operatorname{Base}(x) = \operatorname{Base}(y) \wedge x \sqsubseteq y)$ for unfixed filters $\mathcal{X}$, $\mathcal{Y}$.

LEMMA 720. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathscr{S}\mathcal{X} \sqsubseteq \mathscr{S}\mathcal{Y}$ for every unfixed filters $\mathcal{X}$, $\mathcal{Y}$.

PROOF.

$\Rightarrow$. Suppose $\mathcal{X} \sqsubseteq \mathcal{Y}$. Then there exist $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\operatorname{Base}(x) = \operatorname{Base}(y)$ and $x \sqsubseteq y$. Then $\mathscr{S}\mathcal{X} = \mathscr{S}x \sqsubseteq \mathscr{S}y = \mathscr{S}\mathcal{Y}$.

$\Leftarrow$. Suppose $\mathscr{S}\mathcal{X} \sqsubseteq \mathscr{S}\mathcal{Y}$. Then there are $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\mathscr{S}x \sqsubseteq \mathscr{S}y$. Consequently $\mathscr{S}x' \sqsubseteq \mathscr{S}y'$ for $x' = x \div (\operatorname{Base}(x) \sqcup \operatorname{Base}(y))$, $y' = y \div (\operatorname{Base}(x) \sqcup \operatorname{Base}(y))$. So we have $x' \in \mathcal{X}$, $y' \in \mathcal{Y}$, $\operatorname{Base}(x') = \operatorname{Base}(y')$ and $x' \sqsubseteq y'$, thus $\mathcal{X} \sqsubseteq \mathcal{Y}$.

□

THEOREM 721. $\sqsubseteq$ on the set of unfixed filters is a poset.

PROOF.

Reflexivity. From the previous theorem.
Transitivity. From the previous theorem.

[3]Use this proposition to shorten proofs of other theorem about equivalence of filters? (Our proof uses transitivity of equivalence of filters. So we can't use it to prove that it is an equivalence relation, to avoid circular proof.)