$\Rightarrow$. Suppose $\mathcal{X} \sim \mathcal{Y}$ that is there exists a set P such that $\mathscr{P}P \cap \mathcal{X} = \mathscr{P}P \cap \mathcal{Y}$ and $P \in \mathcal{X}$, $P \in \mathcal{Y}$. Then $\mathcal{X} \div \operatorname{Base}(\mathcal{Y}) = (\mathscr{P}P \cap \mathcal{X}) \cup \left\{ \frac{K \in \mathscr{P} \operatorname{Base}(\mathcal{Y})}{K \sqsupseteq P} \right\} = (\mathscr{P}P \cap \mathcal{Y}) \cup \left\{ \frac{K \in \mathscr{P} \operatorname{Base}(\mathcal{Y})}{K \sqsupseteq P} \right\} = \mathcal{Y}$. So $\mathcal{X} \div \operatorname{Base}(\mathcal{Y}) = \mathcal{Y}$, $\mathcal{Y} \div \operatorname{Base}(\mathcal{X}) = \mathcal{X}$ is similar.

$\Leftarrow$. If $\operatorname{Base}(\mathcal{X}) \notin \mathcal{Y}$ then $\mathcal{Y} \div \operatorname{Base}(\mathcal{X}) \not\ni \operatorname{Base}(\mathcal{Y}) \in \mathcal{Y}$ and thus $\mathcal{Y} \div \operatorname{Base}(\mathcal{X}) \neq \mathcal{Y}$.

So $\operatorname{Base}(\mathcal{X}) \in \mathcal{Y}$ and similarly $\operatorname{Base}(\mathcal{Y}) \in \mathcal{X}$. Thus $\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}) \in \mathcal{Y}$ and similarly $\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}) \in \mathcal{X}$.

It's enough to show $\mathcal{X} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y})) = \mathcal{Y} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}))$ because for every $P \in \mathcal{X}, \mathcal{Y}$ we have $\mathcal{X} \cap \mathscr{P}P = \mathcal{X} \div P = (\mathcal{X} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}))) \div P$ and similarly $\mathcal{Y} \cap \mathscr{P}P = (\mathcal{Y} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}))) \div P$. But it follows from the conditions and proposition 705.

□

Proposition 710. If two filters with the same base are equivalent they are equal.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ be two filters and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$ for some set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$, and $\operatorname{Base}(\mathcal{A}) = \operatorname{Base}(\mathcal{B})$. Then

$$\mathcal{A} = (\mathscr{P}X \cap \mathcal{A}) \cup \left\{ \frac{Y \in D \operatorname{Base}(\mathcal{A})}{Y \sqsupseteq X} \right\} = (\mathscr{P}X \cap \mathcal{B}) \cup \left\{ \frac{Y \in D \operatorname{Base}(\mathcal{B})}{Y \sqsupseteq X} \right\} = \mathcal{B}.$$

□

Proposition 711. If $A \in \mathscr{S}\mathcal{A}$ then $\mathcal{A} \div A \sim \mathcal{A}$.

Proof.

$$(\mathcal{A} \div A) \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})) = \mathscr{S}\mathcal{A} \cap \mathscr{P}A \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})) = \mathscr{S}\mathcal{A} \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})) = \mathcal{A} \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})).$$

Thus $\mathcal{A} \div A \sim \mathcal{A}$ because $A \sqcap \operatorname{Base}(\mathcal{A}) \sqsupseteq X \in \mathcal{A}$ for some $X \in \mathcal{A}$ and

$$A \sqcap \operatorname{Base}(\mathcal{A}) \sqsupseteq X \sqcap \operatorname{Base}(\mathcal{A}) \in \mathcal{A} \div A.$$

□

Proposition 712. $\sim$ is an equivalence relation.

Proof.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ for some filters $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Then there exist a set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$ and a set Y such that $Y \in \mathcal{B}$ and $Y \in \mathcal{C}$ and $\mathscr{P}Y \cap \mathcal{B} = \mathscr{P}Y \cap \mathcal{C}$. So $X \sqcap Y \in \mathcal{A}$ because

$$\mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{B} = \mathscr{P}(X \sqcap Y) \cap \mathcal{B} \supseteq \{X \sqcap Y\} \cap \mathcal{B} \ni X \sqcap Y.$$

Similarly we have $X \sqcap Y \in \mathcal{C}$. Finally

$$\mathscr{P}(X \sqcap Y) \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{B} = \mathscr{P}X \cap \mathscr{P}Y \cap \mathcal{B} = \mathscr{P}X \cap \mathscr{P}Y \cap \mathcal{C} = \mathscr{P}(X \sqcap Y) \cap \mathcal{C}.$$

□