The following is an embedding from filters $\mathcal{A}$ on a lattice DA into the lattice of filters on $\mathfrak{Z}$: $\mathscr{S}\mathcal{A} = \left\{ \frac{K \in \mathfrak{Z}}{\exists X \in \mathcal{A} : X \sqsubseteq K} \right\}$.

PROPOSITION 700. Values of this embedding are filters on the lattice $\mathfrak{Z}$.

PROOF. That $\mathscr{S}\mathcal{A}$ is an upper set is obvious.

Let $P, Q \in \mathscr{S}\mathcal{A}$. Then $P, Q \in \mathfrak{Z}$ and there is an $X \in \mathcal{A}$ such that $X \sqsubseteq P$ and $Y \in \mathcal{A}$ such that $Y \sqsubseteq Q$. So $X \sqcap Y \in \mathcal{A}$ and $P \sqcap Q \sqsupseteq X \sqcap Y \in \mathcal{A}$, so $P \sqcap Q \in \mathscr{S}A$. □

5.39.1. Rebase of filters.

DEFINITION 701. *Rebase* for every filter $\mathcal{A}$ and every $A \in \mathfrak{Z}$ is $\mathcal{A} \div A = \prod \left\{ \frac{\uparrow^A (X \sqcap A)}{X \in \mathcal{A}} \right\}$.

OBVIOUS 702. $\langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}$ is a filter on A.

PROPOSITION 703. The rebase conforms to the formula

$$\mathcal{A} \div A = \langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}.$$

PROOF. We know that $\langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}$ is a filter.

If $P \in \langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}$ then $P \in \mathscr{P}A$ and $Y \sqcap A \sqsubseteq P$ for some $Y \in \mathcal{A}$. Thus $P \sqsupseteq Y \sqcap A \in \prod \left\{ \frac{\uparrow^A (Y \sqcap A)}{Y \in \mathcal{A}} \right\}$.

If $P \in \prod \left\{ \frac{\uparrow^A (X \sqcap A)}{X \in \mathcal{A}} \right\}$ then by properties of generalized filter bases, there exists $X \in \mathcal{A}$ such that $P \sqsupseteq X \sqcap A$. Also $P \in \mathscr{P}A$. Thus $P \in \langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}$. □

PROPOSITION 704. $\mathcal{X} \div \operatorname{Base}(\mathcal{X}) = \mathcal{X}$.

PROOF. Because $X \sqcap \operatorname{Base}(\mathcal{X}) = X$ for $X \in \mathcal{X}$. □

PROPOSITION 705. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \sqsubseteq A$.

PROOF. $(\mathcal{X} \div A) \div B = \prod \left\{ \frac{\uparrow^B (Y \sqcap B)}{Y \in \prod \left\{ \frac{\uparrow^A (X \sqcap A)}{X \in \mathcal{X}} \right\}} \right\} = \prod \left\{ \frac{\uparrow^B (X \sqcap A)}{X \in \mathcal{X}} \right\} \sqcap \uparrow^B B = \prod \left\{ \frac{\uparrow^B (X \sqcap A \sqcap B)}{X \in \mathcal{X}} \right\} = \prod \left\{ \frac{\uparrow^B (X \sqcap B)}{X \in \mathcal{X}} \right\} = \mathcal{X} \div B$. □

PROPOSITION 706. If $A \in \mathcal{A}$ then $\mathcal{A} \div A = \mathcal{A} \cap \mathscr{P}A$.

PROOF. $\mathcal{A} \div A = \langle A\sqcap\rangle^* \mathscr{S}\mathcal{A} = \langle A\sqcap\rangle^* \left\{ \frac{K \in \mathfrak{Z}}{\exists X \in \mathcal{A} : X \sqsubseteq K} \right\} = \left\{ \frac{K \in \mathfrak{Z}}{K \in \mathcal{A} \wedge K \in \mathscr{P}A} \right\} = \mathcal{A} \cap \mathscr{P}A$. □

PROPOSITION 707. Let filters $\mathcal{X}$ and $\mathcal{Y}$ be such that $\operatorname{Base}(\mathcal{X}) = \operatorname{Base}(\mathcal{Y}) = B$. Then $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} = \mathcal{Y}$ for every $\mathfrak{Z} \ni C \sqsupseteq B$.

PROOF. $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} = \mathcal{Y} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} \Leftrightarrow \mathcal{X} = \mathcal{Y}$. □

5.39.2. Equivalence of filters.

DEFINITION 708. Two filters $\mathcal{A}$ and $\mathcal{B}$ (with possibly different base sets) are equivalent ($\mathcal{A} \sim \mathcal{B}$) iff there exists an $X \in \mathfrak{Z}$ such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$.

PROPOSITION 709. $\mathcal{X}$ and $\mathcal{Y}$ are equivalent iff ($\mathcal{X} \sim \mathcal{Y}$) iff $\mathcal{Y} = \mathcal{X} \div \operatorname{Base}(\mathcal{Y})$ and $\mathcal{X} = \mathcal{Y} \div \operatorname{Base}(\mathcal{X})$.

PROOF.