PROOF. $\exists \mathcal{F} \in K : A \in \operatorname{up} \mathcal{F}$ for every infinite set A.

The set A can be partitioned into two infinite sets A_1, A_2.

Take $\mathcal{F}_1, \mathcal{F}_2 \in K$ such that $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$.

$\mathcal{F}_1 \neq \mathcal{F}_2$ because otherwise A_1 and A_2 are not disjoint.

Obviously $A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$.

So there exist two different $\mathcal{F} \in K$ such that $A \in \operatorname{up} \mathcal{F}$. Consequently $\exists \mathcal{F} \in K \setminus \{\mathcal{G}\} : A \in \operatorname{up} \mathcal{F}$ that is $\bigsqcup (K \setminus \{\mathcal{G}\}) = \Omega$. □

EXAMPLE 698. There exists a filter on a set which cannot be weakly partitioned into ultrafilters.

PROOF. Consider cofinite filter Ω on any infinite set.

Suppose K is its weak partition into ultrafilters. Then $x \asymp \bigsqcup (K \setminus \{x\})$ for some ultrafilter $x \in K$.

We have $\bigsqcup (K \setminus \{x\}) \sqsubset \bigsqcup K$ (otherwise $x \sqsubseteq \bigsqcup (K \setminus \{x\})$) what is impossible due the last lemma. □

COROLLARY 699. There exists a filter on a set which cannot be strongly partitioned into ultrafilters.

5.37. Open problems about filters

Under which conditions $a \setminus^* b$ and $a \# b$ are complementive to a?

Generalize straight maps for arbitrary posets.

5.38. Further notation

Below to define funcoids and reloids we need a fixed powerset filtrator.

Let $(\mathscr{F} A, \mathscr{T} A)$ be an arbitrary but fixed powerset filtrator. This filtrator exists by the theorem 459.

I will call elements of $\mathscr{F}$ *filter objects*.

For brevity we will denote lattice operations on $\mathscr{F} A$ without indexes (for example, take $\bigsqcap S = \bigsqcap^{\mathscr{F} A} S$ for $S \in \mathscr{P} \mathscr{F} A$).

Note that above we also took operations on $\mathscr{T} A$ without indexes (for example, take $\bigsqcap S = \bigsqcap^{\mathscr{T} A} S$ for $S \in \mathscr{P} \mathscr{T} A$).

Because we identify $\mathscr{T} A$ with principal elements of $\mathscr{F} A$, the notation like $\bigsqcap S$ for $S \in \mathscr{P} \mathscr{T} A$ would be inconsistent (it can mean both $\bigsqcap^{\mathscr{T} A} S$ or $\bigsqcap^{\mathscr{F} A} S$). We explicitly state that $\bigsqcap S$ in this case does *not* mean $\bigsqcap^{\mathscr{F} A} S$.

For $\mathcal{X} \in \mathscr{F}$ we will denote $\operatorname{GR} \mathcal{X}$ the corresponding filter on $\mathscr{P} A$. It is a convenient notation to describe relations between filters and sets, consider for example the formula: $\{x\} \subseteq \bigcap \operatorname{GR} \mathcal{X}$.

We will denote lattice operations without pointing a specific set like $\bigsqcap^{\mathscr{F}} S = \bigsqcap^{\mathscr{F}(A)} S$ for a set $S \in \mathscr{P} \mathscr{F}(A)$.

5.39. Equivalent filters and rebase of filters

FiXme: This section was checked for errors but less carefully than the rest of the book.

Throughout this section we will assume that $\mathfrak{Z}$ is a lattice.

An important example: $\mathfrak{Z}$ is the lattice of all small (regarding some Grothendieck universe) sets. (This $\mathfrak{Z}$ is not a powerset, and even not a complete lattice.)

Throughout this section I will use the word *filter* to denote a filter on a sublattice DA where $A \in \mathfrak{Z}$ (if not told explicitly to be a filter on some other set).