

EXAMPLE 689. There are tornings which are not weak partitions.

PROOF.  $\{\frac{\Delta+a}{a \in \mathbb{R}}\}$  is a torning but not weak partition of the real line.  $\square$

LEMMA 690. Let  $\mathfrak{A}$  be the set of filters on a set  $U$ . Then  $X \sqcap^{\mathfrak{A}} \Omega \sqsubseteq Y \sqcap^{\mathfrak{A}} \Omega$  iff  $X \setminus Y$  is a finite set, having fixed sets  $X, Y \in \mathcal{P}U$ .

PROOF. Let  $M$  be the set of finite subsets of  $U$ .

$$\begin{aligned} X \sqcap^{\mathfrak{A}} \Omega \sqsubseteq Y \sqcap^{\mathfrak{A}} \Omega &\Leftrightarrow \\ \left\{ \frac{X \cap K_X}{K_X \in \Omega} \right\} \supseteq \left\{ \frac{Y \cap K_Y}{K_Y \in \Omega} \right\} &\Leftrightarrow \\ \forall K_Y \in \Omega \exists K_X \in \Omega : Y \cap K_Y = X \cap K_X &\Leftrightarrow \\ \forall L_Y \in M \exists L_X \in M : Y \setminus L_Y = X \setminus L_X &\Leftrightarrow \\ \forall L_Y \in M : X \setminus (Y \setminus L_Y) \in M &\Leftrightarrow \\ X \setminus Y \in M. & \end{aligned}$$

$\square$

EXAMPLE 691. There exists a filter  $\mathcal{A}$  on a set  $U$  such that  $(\mathcal{P}U)/\sim$  and  $Z(D\mathcal{A})$  are not complete lattices.

PROOF. Due to the isomorphism it is enough to prove for  $(\mathcal{P}U)/\sim$ .

Let take  $U = \mathbb{N}$  and  $\mathcal{A} = \Omega$  be the Fréchet filter on  $\mathbb{N}$ .

Partition  $\mathbb{N}$  into infinitely many infinite sets  $A_0, A_1, \dots$ . To withhold our example we will prove that the set  $\{[A_0], [A_1], \dots\}$  has no supremum in  $(\mathcal{P}U)/\sim$ .

Let  $[X]$  be an upper bound of  $[A_0], [A_1], \dots$  that is  $\forall i \in \mathbb{N} : X \sqcap^{\Omega} \Omega \supseteq A_i \sqcap^{\Omega} \Omega$  that is  $A_i \setminus X$  is finite. Consequently  $X$  is infinite. So  $X \cap A_i \neq \emptyset$ .

Choose for every  $i \in \mathbb{N}$  some  $z_i \in X \cap A_i$ . The  $\{z_0, z_1, \dots\}$  is an infinite subset of  $X$  (take into account that  $z_i \neq z_j$  for  $i \neq j$ ). Let  $Y = X \setminus \{z_0, z_1, \dots\}$ . Then  $Y \sqcap^{\Omega} \Omega \supseteq A_i \sqcap^{\Omega} \Omega$  because  $A_i \setminus Y = A_i \setminus (X \setminus \{z_i\}) = (A_i \setminus X) \cup \{z_i\}$  which is finite because  $A_i \setminus X$  is finite. Thus  $[Y]$  is an upper bound for  $\{[A_0], [A_1], \dots\}$ .

Suppose  $Y \sqcap^{\Omega} \Omega = X \sqcap^{\Omega} \Omega$ . Then  $Y \setminus X$  is finite what is not true. So  $Y \sqcap^{\Omega} \Omega \sqsubset X \sqcap^{\Omega} \Omega$  that is  $[Y]$  is below  $[X]$ .  $\square$

### 5.36.1. Weak and Strong Partition.

DEFINITION 692. A family  $S$  of subsets of a countable set is *independent* iff the intersection of any finitely many members of  $S$  and the complements of any other finitely many members of  $S$  is infinite.

LEMMA 693. The “infinite” at the end of the definition could be equivalently replaced with “nonempty” if we assume that  $S$  is infinite.

PROOF. Suppose that some sets from the above definition has a finite intersection  $J$  of cardinality  $n$ . Then (thanks  $S$  is infinite) get one more set  $X \in S$  and we have  $J \cap X \neq \emptyset$  and  $J \cap (\mathbb{N} \setminus X) \neq \emptyset$ . So  $\text{card}(J \cap X) < n$ . Repeating this, we prove that for some finite family of sets we have empty intersection what is a contradiction.  $\square$

LEMMA 694. There exists an independent family on  $\mathbb{N}$  of cardinality  $\mathfrak{c}$ .

PROOF. Let  $C$  be the set of finite subsets of  $\mathbb{Q}$ . Since  $\text{card } C = \text{card } \mathbb{N}$ , it suffices to find  $\mathfrak{c}$  independent subsets of  $C$ . For each  $r \in \mathbb{R}$  let

$$E_r = \left\{ \frac{F \in C}{\text{card}(F \cap \cdot) - \infty; r \text{ is even}} \right\}.$$