Proof. $\partial\Omega(U) = \neg\langle\neg\rangle^*\Omega(U)$.

$\langle\neg\rangle^*\Omega$ is the set of finite subsets of U. Thus $\neg\langle\neg\rangle^*\Omega(U)$ is the set of infinite subsets of U. □

5.34.2. Number of Filters on a Set.

Definition 671. A collection Y of sets has finite intersection property iff intersection of any finite subcollection of Y is non-empty.

The following was borrowed from [7]. Thanks to Andreas Blass for email support about his proof.

Lemma 672. (by Hausdorff) For an infinite set X there is a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that given any disjoint finite subfamilies $\mathcal{A}$, $\mathcal{B}$, the intersection of sets in $\mathcal{A}$ and complements of sets in $\mathcal{B}$ is nonempty.

Proof. Let

$$X' = \left\{ \frac{(P,Q)}{P \in \mathscr{P}X \text{ is finite}, Q \in \mathscr{P}\mathscr{P}P} \right\}.$$

It's easy to show that $\operatorname{card} X' = \operatorname{card} X$. So it is enough to show this for X' instead of X. Let

$$\mathcal{F} = \left\{ \frac{\left\{ \frac{(P,Q)\in X'}{Y\cap P\in Q} \right\}}{Y \in \mathscr{P}X} \right\}.$$

To finish the proof we show that for every disjoint finite $Y_+ \in \mathscr{P}\mathscr{P}X$ and finite $Y_- \in \mathscr{P}\mathscr{P}X$ there exist $(P,Q) \in X'$ such that

$$\forall Y \in Y_+ : (P,Q) \in \left\{ \frac{(P,Q) \in X'}{Y \cap P \in Q} \right\} \quad \text{and} \quad \forall Y \in Y_- : (P,Q) \notin \left\{ \frac{(P,Q) \in X'}{Y \cap P \in Q} \right\}$$

what is equivalent to existence $(P,Q) \in X'$ such that

$$\forall Y \in Y_+ : Y \cap P \in Q \quad \text{and} \quad \forall Y \in Y_- : Y \cap P \notin Q.$$

For existence of this (P,Q), it is enough existence of P such that intersections $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$.

Really, for each pair of distinct $Y_0, Y_1 \in Y_+ \cup Y_-$ choose a point which lies in one of the sets Y_0, Y_1 and not in an other, and call the set of such points P. Then $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$. □

Corollary 673. For an infinite set X there is a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that for arbitrary disjoint subfamilies $\mathcal{A}$ and $\mathcal{B}$ the set $\mathcal{A} \cup \left\{ \frac{X\setminus A}{A \in \mathcal{B}} \right\}$ has finite intersection property.

Theorem 674. Let X be a set. The number of ultrafilters on X is $2^{2^{\operatorname{card} X}}$ if X is infinite and $\operatorname{card} X$ if X is finite.

Proof. The finite case follows from the fact that every ultrafilter on a finite set is trivial. Let X be infinite. From the lemma, there exists a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that for every $\mathcal{G} \in \mathscr{P}\mathcal{F}$ we have $\Phi(\mathcal{F},\mathcal{G}) = \bigsqcap^{\mathfrak{A}} \mathcal{G} \sqcap \bigsqcap^{\mathfrak{A}} \left\{ \frac{X\setminus A}{A \in \mathcal{F}\setminus\mathcal{G}} \right\} \neq \bot^{\mathfrak{A}(X)}$.

This filter contains all sets from $\mathcal{G}$ and does not contain any sets from $\mathcal{F} \setminus \mathcal{G}$. So for every suitable pairs $(\mathcal{F}_0, \mathcal{G}_0)$ and $(\mathcal{F}_1, \mathcal{G}_1)$ there is $A \in \Phi(\mathcal{F}_0, \mathcal{G}_0)$ such that $\overline{A} \in \Phi(\mathcal{F}_1, \mathcal{G}_1)$. Consequently all filters $\Phi(\mathcal{F},\mathcal{G})$ are disjoint. So for every pair $(\mathcal{F},\mathcal{G})$ where $\mathcal{G} \in \mathscr{P}\mathcal{F}$ there exist a distinct ultrafilter under $\Phi(\mathcal{F},\mathcal{G})$, but the number of such pairs $(\mathcal{F},\mathcal{G})$ is $2^{2^{\operatorname{card} X}}$. Obviously the number of all filters is not above $2^{2^{\operatorname{card} X}}$. □