5.34.1. Fréchet Filter.

DEFINITION 662. $\Omega = \left\{ \frac{\mathfrak{U} \setminus X}{X \text{ is a finite subset of } \mathfrak{U}} \right\}$ is called either *Fréchet filter* or *cofinite filter*.

It is trivial that Fréchet filter is a filter.

PROPOSITION 663. $\operatorname{Cor} \Omega = \bot^3$; $\bigcap \Omega = \emptyset$.

PROOF. This can be deduced from the formula $\forall \alpha \in \mathfrak{U} \exists X \in \Omega : \alpha \notin X$. □

THEOREM 664. $\max\left\{ \frac{\mathcal{X} \in \mathfrak{A}}{\operatorname{Cor} \mathcal{X} = \bot^3} \right\} = \max\left\{ \frac{\mathcal{X} \in \mathfrak{A}}{\bigcap \mathcal{X} = \emptyset} \right\} = \Omega$.

PROOF. Due the last proposition, it is enough to show that $\operatorname{Cor} \mathcal{X} = \bot^3 \Rightarrow \mathcal{X} \sqsubseteq \Omega$ for every filter $\mathcal{X}$.

Let $\operatorname{Cor} \mathcal{X} = \bot^3$ for some filter $\mathcal{X}$. Let $X \in \Omega$. We need to prove that $X \in \mathcal{X}$. $X = \mathfrak{U} \setminus \{\alpha_0, \ldots, \alpha_n\}$. $\mathfrak{U} \setminus \{\alpha_i\} \in \mathcal{X}$ because otherwise $\alpha_i \in \uparrow^{-1} \operatorname{Cor} \mathcal{X}$. So $X \in \mathcal{X}$. □

THEOREM 665. $\Omega = \bigsqcup^{\mathfrak{A}} \left\{ \frac{x}{x \text{ is a non-trivial ultrafilter}} \right\}$.

PROOF. It follows from the facts that $\operatorname{Cor} x = \bot^3$ for every non-trivial ultrafilter x, that $\mathfrak{A}$ is an atomistic lattice, and the previous theorem. □

THEOREM 666. Cor is the lower adjoint of $\Omega \sqcup^{\mathfrak{A}} -$.

PROOF. Because both Cor and $\Omega \sqcup^{\mathfrak{A}} -$ are monotone, it is enough (theorem 126) to prove (for every filters $\mathcal{X}$ and $\mathcal{Y}$)

$$\mathcal{X} \sqsubseteq \Omega \sqcup^{\mathfrak{A}} \operatorname{Cor} \mathcal{X} \quad \text{and} \quad \operatorname{Cor}(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) \sqsubseteq \mathcal{Y}.$$

$\operatorname{Cor}(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) = \operatorname{Cor} \Omega \sqcup^3 \operatorname{Cor} \mathcal{Y} = \bot^3 \sqcup^3 \operatorname{Cor} \mathcal{Y} = \operatorname{Cor} \mathcal{Y} \sqsubseteq \mathcal{Y}$.
$\Omega \sqcup^{\mathfrak{A}} \operatorname{Cor} \mathcal{X} \sqsupseteq \operatorname{Edg} \mathcal{X} \sqcup^{\mathfrak{A}} \operatorname{Cor} \mathcal{X} = \mathcal{X}$. □

COROLLARY 667. $\operatorname{Cor} \mathcal{X} = \mathcal{X} \setminus^* \Omega$ for every filter on a set.

PROOF. By theorem 154. □

COROLLARY 668. $\operatorname{Cor} \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} \langle \operatorname{Cor} \rangle^* S$ for any set S of filters on a powerset.

This corollary can be rewritten in elementary terms and proved elementarily:

PROPOSITION 669. $\bigcap \bigcap S = \bigcup_{F \in S} \bigcap F$ for a set S of filters on some set.

PROOF. (by ANDREAS BLASS) The $\supseteq$ direction is rather formal. Consider any one of the sets being intersected on the left side, i.e., any set X that is in all the filters in S, and consider any of the sets being unioned (that's not a word, but you know what I mean) on the right, i.e., $\bigcap F$ for some $F \in S$. Then, since $X \in F$, we have $\bigcap F \subseteq X$. Taking the union over all $F \in S$ (while keeping X fixed), we get that the right side of your equation is $\subseteq X$. Since that's true for all $X \in \bigcap S$, we infer that the right side is a subset of the left side. (This argument seems to work in much greater generality; you just need that the relevant infima (in place of intersections) exist in your poset.)

For the $\subseteq$ direction, consider any element $x \in \bigcap \bigcap S$, and suppose, toward a contradiction, that it is not an element of the union on the right side of your equation. So, for each $F \in S$, we have $x \notin \bigcap F$, and therefore we can find a set $A_F \in F$ with $x \notin A_F$. Let $B = \bigcup_{F \in S} A_F$ and notice that $B \in F$ for every $F \in S$ (because $B \supseteq A_F$). So $B \in \bigcap S$. But, by choice of the A_F's, we have $x \notin B$, contrary to the assumption that $x \in \bigcap \bigcap S$. □

PROPOSITION 670. $\partial \Omega(U)$ is the set of infinite subsets of U.