PROOF. We need to prove $(\lambda i \in n : a_i \setminus b_i) \sqcap b = \bot$ and $a \sqcup b = b \sqcup (\lambda i \in n : a_i \setminus b_i)$. Really

$$(\lambda i \in n : a_i \setminus b_i) \sqcap b = \lambda i \in n : (a_i \setminus b_i) \sqcap b_i = \bot;$$
$$b \sqcup (\lambda i \in n : a_i \setminus b_i) = \lambda i \in n : b_i \sqcup (a_i \setminus b_i) = \lambda i \in n : b_i \sqcup a_i = a \sqcup b.$$

□

PROPOSITION 658. If every $\mathfrak{A}_i$ is a distributive lattice, then $a \setminus^* b = \lambda i \in \operatorname{dom} \mathfrak{A} : a_i \setminus^* b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \setminus^* b_i$ is defined.

PROOF. We need to prove that $\lambda i \in \operatorname{dom} \mathfrak{A} : a_i \setminus^* b_i = \bigsqcap \left\{ \frac{z \in \prod \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$.

To prove it is enough to show $a_i \setminus^* b_i = \bigsqcap \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, a \sqsubseteq b \sqcup z} \right\}$ that is $a_i \setminus^* b_i = \bigsqcap \left\{ \frac{z \in \mathfrak{A}_i}{a_i \sqsubseteq b_i \sqcup z} \right\}$ because $z' \in \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, a \sqsubseteq b \sqcup z} \right\} \Leftrightarrow z' \in \left\{ \frac{z \in \mathfrak{A}_i}{a_i \sqsubseteq b_i \sqcup z} \right\}$ (for the reverse implication take $z_j = a_i$ for $j \neq i$), but $a_i \setminus^* b_i = \bigsqcap \left\{ \frac{z \in \mathfrak{A}_i}{a_i \sqsubseteq b_i \sqcup z} \right\}$ is true by definition. □

PROPOSITION 659. If every $\mathfrak{A}_i$ is a distributive lattice with least element, then $a \# b = \lambda i \in \operatorname{dom} \mathfrak{A} : a_i \# b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \# b_i$ is defined.

PROOF. We need to prove that $\lambda i \in \operatorname{dom} \mathfrak{A} : a_i \# b_i = \bigsqcup \left\{ \frac{z \in \prod \mathfrak{A}}{z \sqsubseteq a \wedge z \asymp b} \right\}$.

To prove it is enough to show $a_i \# b_i = \bigsqcup \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, z \sqsubseteq a \wedge z \asymp b} \right\}$ that is $a_i \# b_i = \bigsqcup \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, z_i \sqsubseteq a_i \wedge \forall j \in \operatorname{dom} \mathfrak{A} : z_j \asymp b_j} \right\}$ that is $a_i \# b_i = \bigsqcup \left\{ \frac{z \in \mathfrak{A}_i}{z \sqsubseteq a_i \wedge z \asymp b_i} \right\}$ (take $z_j = \bot^{\mathfrak{A}_j}$ for $j \neq i$) what is true by definition. □

PROPOSITION 660. Let every $\mathfrak{A}_i$ be a poset with least element and a_i^* is defined. Then $a^* = \lambda i \in \operatorname{dom} \mathfrak{A} : a_i^*$.

PROOF. We need to prove that $\lambda i \in \operatorname{dom} \mathfrak{A} : a_i^* = \bigsqcup \left\{ \frac{c \in \prod \mathfrak{A}}{c \asymp a} \right\}$. To prove this it is enough to show that $a_i^* = \bigsqcup \left\{ \frac{c_i}{c \in \prod \mathfrak{A}, c \asymp a} \right\}$ that is $a_i^* = \bigsqcup \left\{ \frac{c_i}{c \in \prod \mathfrak{A}, \forall j \in \operatorname{dom} \mathfrak{A} : c_j \asymp a_j} \right\}$ that is $a_i^* = \bigsqcup \left\{ \frac{c_i}{c \in \prod \mathfrak{A}, c_i \asymp a_i} \right\}$ (take $c_j = \bot^{\mathfrak{A}_j}$ for $j \neq i$) that is $a_i^* = \bigsqcup \left\{ \frac{c \in \mathfrak{A}_i}{c \asymp a_i} \right\}$ what is true by definition. □

COROLLARY 661. Let every $\mathfrak{A}_i$ be a poset with greatest element and a_i^+ is defined. Then $a^+ = \lambda i \in \operatorname{dom} \mathfrak{A} : a_i^+$.

PROOF. By duality. □

5.34. Filters on a Set

In this section we will fix a powerset filtrator $(\mathfrak{A}, \mathfrak{Z}) = (\mathfrak{A}, \mathscr{P}\mathfrak{U})$ for some set $\mathfrak{U}$.

The consideration below is about filters on a set $\mathfrak{U}$, but this can be generalized for filters on complete atomic boolean algebras due complete atomic boolean algebras are isomorphic to algebras of sets on some set $\mathfrak{U}$.