PROPOSITION 648. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ where $i \in n$ (for some index set n) is a down-aligned filtrator with separable core then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is with separable core.

PROOF. Let $a \neq b$. Then $\exists i \in n : a_i \neq b_i$. So $\exists x \in \mathfrak{Z}_i : (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = \lambda j \in n : \begin{cases} x & \text{if } j = i \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases}$. Then we have $y \not\asymp a$ and $y \asymp b$ and $y \in \mathfrak{Z}$. □

PROPOSITION 649. Let every $\mathfrak{A}_i$ be a bounded lattice. Every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a central filtrator iff $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a central filtrator.

PROOF.

$$\begin{aligned} x \in Z\left(\prod \mathfrak{A}\right) \Leftrightarrow \\ \exists y \in \prod \mathfrak{A} : (x \sqcap y = \bot^{\prod \mathfrak{A}} \wedge x \sqcup y = \top^{\prod \mathfrak{A}}) \Leftrightarrow \\ \exists y \in \prod \mathfrak{A} \forall i \in \operatorname{dom} \mathfrak{A} : (x_i \sqcap y_i = \bot^{\mathfrak{A}_i} \wedge x_i \sqcup y_i = \top^{\mathfrak{A}_i}) \Leftrightarrow \\ \forall i \in \operatorname{dom} \mathfrak{A} \exists y \in \mathfrak{A}_i : (x_i \sqcap y = \bot^{\mathfrak{A}_i} \wedge x_i \sqcup y = \top^{\mathfrak{A}_i}) \Leftrightarrow \\ \forall i \in \operatorname{dom} \mathfrak{A} : x_i \in Z(\mathfrak{A}_i). \end{aligned}$$

So

$$Z\left(\prod \mathfrak{A}\right) = \prod \mathfrak{Z} \Leftrightarrow \prod_{i \in \operatorname{dom} \mathfrak{A}} Z(\mathfrak{A}_i) = \prod \mathfrak{Z} \Leftrightarrow$$
$$\text{(because every } \mathfrak{Z}_i \text{ is nonempty)} \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A} : Z(\mathfrak{A}_i) = \mathfrak{Z}_i.$$

□

PROPOSITION 650. For every element a of a product filtrator $(\prod \mathfrak{A}, \prod \mathfrak{Z})$:

1°. $\operatorname{up} a = \prod_{i \in \operatorname{dom} a} \operatorname{up} a_i$;
2°. $\operatorname{down} a = \prod_{i \in \operatorname{dom} a} \operatorname{down} a_i$.

PROOF. We will prove only the first as the second is dual.

$$\operatorname{up} a = \left\{ \frac{c \in \prod \mathfrak{Z}}{c \sqsupseteq a} \right\} = \left\{ \frac{c \in \prod \mathfrak{Z}}{\forall i \in \operatorname{dom} a : c_i \sqsupseteq a_i} \right\} = \left\{ \frac{c \in \prod \mathfrak{Z}}{\forall i \in \operatorname{dom} a : c_i \in \operatorname{up} a_i} \right\} = \prod_{i \in \operatorname{dom} a} \operatorname{up} a_i.$$

□

PROPOSITION 651. If every $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ is a prefiltered filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a prefiltered filtrator.

PROOF. Let $a, b \in \prod \mathfrak{A}$ and $a \neq b$. Then there exists $i \in n$ such that $a_i \neq b_i$ and so $\operatorname{up} a_i \neq \operatorname{up} b_i$. Consequently $\prod_{i \in \operatorname{dom} a} \operatorname{up} a_i \neq \prod_{i \in \operatorname{dom} a} \operatorname{up} b_i$ that is $\operatorname{up} a \neq \operatorname{up} b$. □

PROPOSITION 652. Let every $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ be a filtered filtrator with $\operatorname{up} x \neq \emptyset$ for every $x \in \mathfrak{A}_i$ (for every $i \in n$). Then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a filtered filtrator.

PROOF. Let every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be a filtered filtrator. Let $\operatorname{up} a \supseteq \operatorname{up} b$ for some $a, b \in \prod \mathfrak{A}$. Then $\prod_{i \in \operatorname{dom} a} \operatorname{up} a_i \supseteq \prod_{i \in \operatorname{dom} a} \operatorname{up} b_i$ and consequently (taking into account that $\operatorname{up} x \neq \emptyset$ for every $x \in \mathfrak{A}_i$) $\operatorname{up} a_i \supseteq \operatorname{up} b_i$ for every $i \in n$. Then $\forall i \in n : a_i \sqsubseteq b_i$ that is $a \sqsubseteq b$. □