Proof. It is enough to prove the formula (2).

It's obvious that $\lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcup Bi \sqsupseteq A, B$.

Let $C \sqsupseteq A, B$. Then (for every $i \in \operatorname{dom}\mathfrak{A}$) $Ci \sqsupseteq Ai$ and $Ci \sqsupseteq Bi$. Thus $Ci \sqsupseteq Ai \sqcup Bi$ that is $C \sqsupseteq \lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcup Bi$. □

Corollary 635. If $\mathfrak{A}_i$ are lattices then $\prod \mathfrak{A}$ is a lattice.

Obvious 636. If $\mathfrak{A}_i$ are distributive lattices then $\prod \mathfrak{A}$ is a distributive lattice.

Proposition 637. If $\mathfrak{A}_i$ are boolean lattices then $\prod \mathfrak{A}$ is a boolean lattice.

Proof. We need to prove only that every element $a \in \prod \mathfrak{A}$ has a complement. But this complement is evidently $\lambda i \in \operatorname{dom} a : \overline{a_i}$. □

Proposition 638. If every $\mathfrak{A}_i$ is a poset then for every $S \in \mathscr{P} \prod \mathfrak{A}$

1°. $\bigsqcup S = \lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcup_{x \in S} x_i$ whenever every $\bigsqcup_{x \in S} x_i$ exists;
2°. $\bigsqcap S = \lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcap_{x \in S} x_i$ whenever every $\bigsqcap_{x \in S} x_i$ exists.

Proof. It's enough to prove the first formula.

$\left(\lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcup_{x \in S} x_i\right)_i = \bigsqcup_{x \in S} x_i \sqsupseteq x_i$ for every $x \in S$ and $i \in \operatorname{dom}\mathfrak{A}$.

Let $y \sqsupseteq x$ for every $x \in S$. Then $y_i \sqsupseteq x_i$ for every $i \in \operatorname{dom}\mathfrak{A}$ and thus $y_i \sqsupseteq \bigsqcup_{x \in S} x_i = \left(\lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcup_{x \in S} x_i\right)_i$ that is $y \sqsupseteq \lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcup_{x \in S} x_i$.

Thus $\bigsqcup S = \lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcup_{x \in S} x_i$ by the definition of join. □

Corollary 639. If $\mathfrak{A}_i$ are posets then for every $S \in \mathscr{P} \prod \mathfrak{A}$

1°. $\bigsqcup S = \lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcup_{x \in S} x_i$ whenever $\bigsqcup S$ exists;
2°. $\bigsqcap S = \lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcap_{x \in S} x_i$ whenever $\bigsqcap S$ exists.

Proof. It is enough to prove that (for every i) $\bigsqcup_{x \in S} x_i$ exists whenever $\bigsqcup S$ exists.

Fix $i \in \operatorname{dom}\mathfrak{A}$.

Take $y_i = \left(\bigsqcup S\right)_i$ and let prove that y_i is the least upper bound of $\left\{ \frac{x_i}{x \in S} \right\}$.

y_i is it's upper bound because $\bigsqcup S \sqsupseteq x$ and thus $\left(\bigsqcup S\right)_i \sqsupseteq x_i$ for every $x \in S$.

Let $x \in S$ and for some $t \in \mathfrak{A}_i$

$$T(t) = \lambda j \in \operatorname{dom}\mathfrak{A} : \begin{cases} t & \text{if } i = j \\ x_j & \text{if } i \neq j. \end{cases}$$

Let $t \sqsupseteq x_i$. Then $T(t) \sqsupseteq x$ for every $x \in S$. So $T(t) \sqsupseteq \bigsqcup S$ and consequently $t = T(t)_i \sqsupseteq y_i$.

So y_i is the least upper bound of $\left\{ \frac{x_i}{x \in S} \right\}$. □

Corollary 640. If $\mathfrak{A}_i$ are complete lattices then $\mathfrak{A}$ is a complete lattice.

Obvious 641. If $\mathfrak{A}_i$ are complete (co-)brouwerian lattices then $\mathfrak{A}$ is a (co-)brouwerian lattice.

Proposition 642. If each $\mathfrak{A}_i$ is a separable poset with least element (for some index set n) then $\prod \mathfrak{A}$ is a separable poset.

Proof. Let $a \neq b$. Then $\exists i \in \operatorname{dom}\mathfrak{A} : a_i \neq b_i$. So $\exists x \in \mathfrak{A}_i : (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = \lambda j \in \operatorname{dom}\mathfrak{A} : \begin{cases} x & \text{if } j = i; \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i. \end{cases}$ Then $y \not\asymp a$ and $y \asymp b$. □