(c) $\bigsqcup(\operatorname{atoms} a \setminus \operatorname{atoms} b)$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. By corollary 528 and theorem 578.
3°⇒4°. Theorem 244.

CONJECTURE 629. $a \setminus^* b = a \# b$ for arbitrary filters a, b on powersets is not provable in ZF (without axiom of choice).

□

5.33. Function spaces of posets

DEFINITION 630. Let $\mathfrak{A}_i$ be a family of posets indexed by some set $\operatorname{dom}\mathfrak{A}$. We will define order of indexed families of elements of posets by the formula

$$a \sqsubseteq b \Leftrightarrow \forall i \in \operatorname{dom}\mathfrak{A} : a_i \sqsubseteq b_i.$$

I will call this new poset $\prod \mathfrak{A}$ *the function space* of posets and the above order *product order.*

PROPOSITION 631. The function space for posets is also a poset.

PROOF.

Reflexivity. Obvious.
Antisymmetry. Obvious.
Transitivity. Obvious.

□

OBVIOUS 632. $\mathfrak{A}$ has least element iff each $\mathfrak{A}_i$ has a least element. In this case

$$\bot^{\prod \mathfrak{A}} = \prod_{i \in \operatorname{dom}\mathfrak{A}} \bot^{\mathfrak{A}_i}.$$

PROPOSITION 633. $a \not\asymp b \Leftrightarrow \exists i \in \operatorname{dom}\mathfrak{A} : a_i \not\asymp b_i$ for every $a, b \in \prod \mathfrak{A}$ if every $\mathfrak{A}_i$ has least element.

PROOF. If $\operatorname{dom}\mathfrak{A} = \emptyset$, then $a = b = \bot$, $a \asymp b$ and thus the theorem statement holds. Assume $\operatorname{dom}\mathfrak{A} \neq \emptyset$.

$$\begin{aligned}
a \not\asymp b &\Leftrightarrow \\
\exists c \in \prod \mathfrak{A} \setminus \{\bot^{\prod \mathfrak{A}}\} : (c \sqsubseteq a \wedge c \sqsubseteq b) &\Leftrightarrow \\
\exists c \in \prod \mathfrak{A} \setminus \{\bot^{\prod \mathfrak{A}}\} \forall i \in \operatorname{dom}\mathfrak{A} : (c_i \sqsubseteq a_i \wedge c_i \sqsubseteq b_i) &\Leftrightarrow \\
\text{(for the reverse implication take } c_j = \bot^{\mathfrak{A}_j} \text{ for } i \neq j) & \\
\exists i \in \operatorname{dom}\mathfrak{A}, c \in \mathfrak{A}_i \setminus \{\bot^{\mathfrak{A}_i}\} : (c \sqsubseteq a_i \wedge c \sqsubseteq b_i) &\Leftrightarrow \\
\exists i \in \operatorname{dom}\mathfrak{A} : a_i \not\asymp b_i. &
\end{aligned}$$

□

PROPOSITION 634.

1°. If $\mathfrak{A}_i$ are join-semilattices then $\mathfrak{A}$ is a join-semilattice and

$$A \sqcup B = \lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcup Bi. \tag{2}$$

2°. If $\mathfrak{A}_i$ are meet-semilattices then $\mathfrak{A}$ is a meet-semilattice and

$$A \sqcap B = \lambda i \in \operatorname{dom}\mathfrak{A} : Ai \sqcap Bi.$$