□

THEOREM 607. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned binarily join-closed and binarily meet-closed distributive lattice filtrator over a boolean lattice.
4°. $A \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} \langle A \sqcap^{\mathfrak{A}} \rangle^* S$ for every $A \in \mathfrak{Z}$ and every set $S \in \mathscr{P}\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. It is binarily join-closed by theorem 531. It is binarily meet-closed by corollary 533. It is distributive by corollary 528.
3°⇒4°. Direct consequence of the lemma.

□

5.27. More about the Lattice of Filters

DEFINITION 608. Atoms of $\mathfrak{F}$ are called *ultrafilters*.

DEFINITION 609. Principal ultrafilters are also called *trivial ultrafilters*.

THEOREM 610. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is central.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. We can conclude that $\mathfrak{A}$ is atomically separable (the corollary 579), with separable core (the theorem 534), and with join-closed core (theorem 531), binarily meet-closed by corollary 533.

We need to prove $Z(\mathfrak{A}) = \mathfrak{Z}$.

Let $\mathcal{X} \in Z(\mathfrak{A})$. Then there exists $\mathcal{Y} \in Z(\mathfrak{A})$ such that $\mathcal{X} \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$ and $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = \top^{\mathfrak{A}}$. Consequently there is $X \in \operatorname{up} \mathcal{X}$ such that $X \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$; we also have $X \sqcup^{\mathfrak{A}} \mathcal{Y} = \top^{\mathfrak{A}}$. Suppose $X \sqsupset \mathcal{X}$. Then there exists $a \in \operatorname{atoms}^{\mathfrak{A}} X$ such that $a \notin \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$. We can conclude also $a \notin \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y}$ (otherwise $X \sqcap^{\mathfrak{A}} \mathcal{Y} \neq \bot^{\mathfrak{A}}$). Thus $a \notin \operatorname{atoms}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y})$ and consequently $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \neq \top^{\mathfrak{A}}$ what is a contradiction. We have $\mathcal{X} = X \in \mathfrak{Z}$.

Let now $X \in \mathfrak{Z}$. Let $Y = \overline{X}$. We have $X \sqcap^{\mathfrak{Z}} Y = \bot^{\mathfrak{A}}$ and $X \sqcup^{\mathfrak{Z}} Y = \top^{\mathfrak{A}}$. Thus $X \sqcap^{\mathfrak{A}} Y = \bigsqcap^{\mathfrak{A}} \{ X \sqcap^{\mathfrak{Z}} Y \} = \bot^{\mathfrak{A}}$; $X \sqcap^{\mathfrak{A}} Y = X \sqcap^{\mathfrak{Z}} Y = \top^{\mathfrak{A}}$. We have shown that $X \in Z(\mathfrak{A})$.

□

5.28. More Criteria

THEOREM 611. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. For every $S \in \mathscr{P}\mathfrak{A}$ the condition $\exists \mathcal{F} \in \mathfrak{A} : S = \star \mathcal{F}$ is equivalent to conjunction of the following items:
   (a) S is a free star on $\mathfrak{A}$;
   (b) S is filter-closed.

PROOF.

1°⇒2°. Obvious.