3°. The three expressions of pseudodifference of a and b in theorem 244 are also equal to $\bigsqcup\left\{\frac{a\sqcap\overline{B}}{B\in\operatorname{up}b}\right\}$.

PROOF.

1°⇒2°. The filtrator of filters on a boolean lattice is:

- complete by corollary 515;
- atomistic by theorem 578;
- co-brouwerian by corollary 528;
- with separable core by theorem 534;
- with binarily meet-closed core by corollary 533.

2°⇒3°. $\bigsqcup\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}\sqsubseteq\bigsqcup\left\{\frac{a\sqcap\overline{B}}{B\in\operatorname{up}b}\right\}$ because

$$z\in\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}\Leftrightarrow z\sqsubseteq a\wedge z\sqcap b=\bot\Leftrightarrow\text{(separability)}$$

$$z\sqsubseteq a\wedge\exists B\in\operatorname{up}b:z\sqcap B=\bot\Leftrightarrow\text{(theorem 601)}\Leftrightarrow z\sqsubseteq a\wedge\exists B\in\operatorname{up}b:z\sqsubseteq\overline{B}\Leftrightarrow$$
$$\exists B\in\operatorname{up}b:\left(z\sqsubseteq a\wedge z\sqsubseteq\overline{B}\right)\Leftrightarrow\exists B\in\operatorname{up}b:z\sqsubseteq a\sqcap\overline{B}\Rightarrow$$
$$z\sqsubseteq\bigsqcup\left\{\frac{a\sqcap\overline{B}}{B\in\operatorname{up}b}\right\}.$$

But $a\sqcap\overline{B}\in\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}$ because

$$\left(a\sqcap\overline{B}\right)\sqcap b=a\sqcap\left(\overline{B}\sqcap b\right)\sqsubseteq a\sqcap\left(\overline{B}\sqcap^{\mathfrak{A}}B\right)=a\sqcap\left(\overline{B}\sqcap^{\mathfrak{Z}}B\right)=a\sqcap\bot=\bot$$

and thus

$$a\sqcap\overline{B}\sqsubseteq\bigsqcup\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}$$

so $\bigsqcup\left\{\frac{z\in\mathscr{F}}{z\sqsubseteq a\wedge z\sqcap b=\bot}\right\}\sqsupseteq\bigsqcup\left\{\frac{a\sqcap\overline{B}}{B\in\operatorname{up}b}\right\}$.

□

5.26. Distributivity for an Element of Boolean Core

LEMMA 606. The following is an implications tuple:

1°. $(\mathfrak{A},\mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A},\mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $(\mathfrak{A},\mathfrak{Z})$ is an up-aligned binarily join-closed and binarily meet-closed distributive lattice filtrator over a boolean lattice.
4°. $A\sqcap^{\mathfrak{A}}$ is a lower adjoint of $\overline{A}\sqcup^{\mathfrak{A}}$ for every $A\in\mathfrak{Z}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is binarily join closed by theorem 531. It is binarily meet-closed by corollary 533. It is distributive by corollary 528.

3°⇒4°. We will use the theorem 126.

That $A\sqcap^{\mathfrak{A}}$ and $\overline{A}\sqcup^{\mathfrak{A}}$ are monotone is obvious.

We need to prove (for every $x,y\in\mathfrak{A}$) that

$$x\sqsubseteq\overline{A}\sqcup^{\mathfrak{A}}\left(A\sqcap^{\mathfrak{A}}x\right)\quad\text{and}\quad A\sqcap^{\mathfrak{A}}\left(\overline{A}\sqcup^{\mathfrak{A}}y\right)\sqsubseteq y.$$

Really,

$$\overline{A}\sqcup^{\mathfrak{A}}(A\sqcap^{\mathfrak{A}}x)=(\overline{A}\sqcup^{\mathfrak{A}}A)\sqcap^{\mathfrak{A}}(\overline{A}\sqcup^{\mathfrak{A}}x)=(\overline{A}\sqcup^{\mathfrak{Z}}A)\sqcap^{\mathfrak{A}}(\overline{A}\sqcup^{\mathfrak{A}}x)=\top\sqcap^{\mathfrak{A}}(\overline{A}\sqcup^{\mathfrak{A}}x)=\overline{A}\sqcup^{\mathfrak{A}}x\sqsupseteq x$$

and

$$A\sqcap^{\mathfrak{A}}(\overline{A}\sqcup^{\mathfrak{A}}y)=(A\sqcap^{\mathfrak{A}}\overline{A})\sqcup^{\mathfrak{A}}(A\sqcap^{\mathfrak{A}}y)=(A\sqcap^{\mathfrak{Z}}\overline{A})\sqcup^{\mathfrak{A}}(A\sqcap^{\mathfrak{A}}y)=\bot\sqcup^{\mathfrak{A}}(A\sqcap^{\mathfrak{A}}y)=A\sqcap^{\mathfrak{A}}y\sqsubseteq y.$$