$\Leftarrow$. We will assume that cardinality of a set is an ordinal defined by von Neumann cardinal assignment (what is a standard practice in ZFC). Recall that $\alpha < \beta \Leftrightarrow \alpha \in \beta$ for ordinals α, β.

We will take it as given that for every nonempty chain $T \in \mathscr{P}S$ we have $\prod T \in S$.

We will prove the following statement: If $\operatorname{card} S = n$ then S is filter closed, for any cardinal n.

Instead we will prove it not only for cardinals but for wider class of ordinals: If $\operatorname{card} S = n$ then S is filter-closed, for any ordinal n.

We will prove it using transfinite induction by n.

For finite n we have $\prod T \in S$ because $T \subseteq S$ has minimal element.

Let $\operatorname{card} T = n$ be an infinite ordinal.

Let the assumption hold for every $m \in \operatorname{card} T$.

We can assign $T = \left\{ \frac{a_\alpha}{\alpha \in \operatorname{card} T} \right\}$ for some a_α because $\operatorname{card} \operatorname{card} T = \operatorname{card} T$.

Consider $\beta \in \operatorname{card} T$.

Let $P_\beta = \left\{ \frac{a_\alpha}{\alpha \in \beta} \right\}$. Let $b_\beta = \prod P_\beta$. Obviously $b_\beta = \prod [P_\beta]_\sqcap$. We have

$$\operatorname{card}[P_\beta]_\sqcap = \operatorname{card} P_\beta = \operatorname{card} \beta < \operatorname{card} T$$

(used the lemma and von Neumann cardinal assignment). By the assumption of induction $b_\beta \in S$.

$\forall \beta \in \operatorname{card} T : P_\beta \subseteq T$ and thus $b_\beta \sqsupseteq \prod T$.

It is easy to see that the set $\left\{ \frac{P_\beta}{\beta \in \operatorname{card} T} \right\}$ is a chain. Consequently $\left\{ \frac{b_\beta}{\beta \in \operatorname{card} T} \right\}$ is a chain.

By the theorem conditions $b = \prod_{\beta \in \operatorname{card} T} b_\beta \in S$ (taken into account that $b_\beta \in S$ by the assumption of induction).

Obviously $b \sqsupseteq \prod T$.

$b \sqsubseteq b_\beta$ and so $\forall \beta \in \operatorname{card} T, \alpha \in \beta : b \sqsubseteq a_\alpha$. Let $\alpha \in \operatorname{card} T$. Then (because $\operatorname{card} T$ is a limit ordinal, see [44]) there exists $\beta \in \operatorname{card} T$ such that $\alpha \in \beta \in \operatorname{card} T$. So $b \sqsubseteq a_\alpha$ for every $\alpha \in \operatorname{card} T$. Thus $b \sqsubseteq \prod T$.

Finally $\prod T = b \in S$.

□

5.20. Co-Separability of Core

THEOREM 587. The following is an implications tuple.

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet infinite distributive complete lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned filtered filtrator whose core is a meet infinite distributive complete lattice.
4°. This filtrator is with co-separable core.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is obviously up-aligned, and filtered by theorem 531.

3°⇒4°. Our filtrator is with join-closed core (theorem 531).

Let $a, b \in \mathfrak{A}$. $\operatorname{Cor} a$ and $\operatorname{Cor} b$ exist since $\mathfrak{Z}$ is a complete lattice.