4°c⇒4°a.

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Rightarrow \exists K \in S : \mathcal{F} \sqcap^{\mathfrak{A}} K \neq \bot \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\mathcal{F} \not\asymp^{\mathfrak{A}} \bigsqcup^{\mathfrak{Z}} S \Rightarrow \exists K \in S : \mathcal{F} \not\asymp^{\mathfrak{A}} K \right) \Leftrightarrow \text{(lemma 551)}$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\overline{\bigsqcup^{\mathfrak{Z}} S} \not\sqsupseteq \mathcal{F} \Rightarrow \exists K \in S : \overline{K} \not\sqsupseteq \mathcal{F} \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\forall K \in S : \overline{K} \sqsupseteq \mathcal{F} \Rightarrow \overline{\bigsqcup^{\mathfrak{Z}} S} \sqsupseteq \mathcal{F} \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\forall K \in S : \overline{K} \sqsupseteq \mathcal{F} \Rightarrow \bigsqcap^{\mathfrak{Z}} \langle \neg \rangle^{*} S \sqsupseteq \mathcal{F} \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\forall K \in S : K \sqsupseteq \mathcal{F} \Rightarrow \bigsqcap^{\mathfrak{Z}} S \sqsupseteq \mathcal{F} \right) \Rightarrow$$

$$\bigsqcap^{\mathfrak{Z}} \operatorname{up} \mathcal{F} \sqsupseteq \mathcal{F} \Leftrightarrow$$

$$\bigsqcap^{\mathfrak{Z}} \operatorname{up} \mathcal{F} \in \operatorname{up} \mathcal{F} \Rightarrow$$

$$\mathcal{F} \in \mathfrak{Z}.$$

□

REMARK 584. The above theorem strengthens theorem 53 in [30]. Both the formulation of the theorem and the proof are considerably simplified.

DEFINITION 585. Let S be a subset of a meet-semilattice. The *filter base generated by* S is the set

$$[S]_{\sqcap} = \left\{ \frac{a_0 \sqcap \cdots \sqcap a_n}{a_i \in S, n = 0, 1, \ldots} \right\}.$$

LEMMA 586. The set of all finite subsets of an infinite set A has the same cardinality as A.

PROOF. Let denote the number of n-element subsets of A as s_n. Obviously $s_n \leq \operatorname{card} A^n = \operatorname{card} A$. Then the number S of all finite subsets of A is equal to

$$s_0 + s_1 + \cdots \leq \operatorname{card} A + \operatorname{card} A + \cdots = \operatorname{card} A.$$

That $S \geq \operatorname{card} A$ is obvious. So $S = \operatorname{card} A$. □

LEMMA 587. A filter base generated by an infinite set has the same cardinality as that set.

PROOF. From the previous lemma. □

DEFINITION 588. Let $\mathfrak{A}$ be a complete lattice. A set $S \in \mathscr{P}\mathfrak{A}$ is *filter-closed* when for every filter base $T \in \mathscr{P}S$ we have $\bigsqcap T \in S$.

THEOREM 589. A subset S of a complete lattice is filter-closed iff for every nonempty chain $T \in \mathscr{P}S$ we have $\bigsqcap T \in S$.

PROOF. (proof sketch by JOEL DAVID HAMKINS)

⇒. Because every nonempty chain is a filter base.