Obviously S' is nonempty and binarily meet-closed. So S' is a generalized filter base. Obviously $\bot^{\mathfrak{A}} \notin S$. So by properties of generalized filter bases $\bigsqcap^{\mathfrak{A}} S' \neq \bot^{\mathfrak{A}}$. But obviously $\bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{A}} S'$. So $\bigsqcap^{\mathfrak{A}} S \neq \bot^{\mathfrak{A}}$. □

COROLLARY 572. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with least element.
3°. Let $S \in \mathscr{P}\mathfrak{Z}$ such that $S \neq \emptyset$ and $A_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} A_n \neq \bot^{\mathfrak{Z}}$ for every $A_0, \ldots, A_n \in S$. Then $\bigsqcap^{\mathfrak{A}} S \neq \bot^{\mathfrak{A}}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Because $(\mathfrak{A}, \mathfrak{Z})$ is binarily meet-closed (by the theorem 532).

□

THEOREM 573. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a bounded meet-semilattice.
3°. $\mathfrak{A}$ is an atomic lattice.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Let $\mathcal{F} \in \mathfrak{A}$. Let choose (by Kuratowski's lemma) a maximal chain S from $\bot^{\mathfrak{A}}$ to $\mathcal{F}$. Let $S' = S \setminus \{\bot^{\mathfrak{A}}\}$. $a = \bigsqcap^{\mathfrak{A}} S' \neq \bot^{\mathfrak{A}}$ by properties of generalized filter bases (the corollary 570 which uses the fact that $\mathfrak{Z}$ is a meet-semilattice with least element). If $a \notin S$ then the chain S can be extended adding there element a because $\bot^{\mathfrak{A}} \sqsubset a \sqsubseteq \mathcal{X}$ for any $\mathcal{X} \in S'$ what contradicts to maximality of the chain. So $a \in S$ and consequently $a \in S'$. Obviously a is the minimal element of S'. Consequently (taking into account maximality of the chain) there is no $\mathcal{Y} \in \mathfrak{A}$ such that $\bot^{\mathfrak{A}} \sqsubset \mathcal{Y} \sqsubset a$. So a is an atomic filter. Obviously $a \sqsubseteq \mathcal{F}$.

□

DEFINITION 574. A complete lattice is *co-compact* iff $\bigsqcap S = \bot$ for a set S of elements of this lattice implies that there is its finite subset $T \subseteq S$ such that $\bigsqcap T = \bot$.

THEOREM 575. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a bounded meet-semilattice.
3°. $\mathfrak{A}$ is co-compact.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Poset $\mathfrak{A}$ is complete by corollary 515.

If $\bot \in \operatorname{up} \bigsqcap^{\mathfrak{A}} S$ then there are $K_i \in \operatorname{up} \bigcup S$ such that $\bot \in \operatorname{up}(K_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} K_n)$ that is $K_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} K_n = \bot$ from which easily follows $\mathcal{F}_0 \sqcap^{\mathfrak{A}} \ldots \sqcap^{\mathfrak{A}} \mathcal{F}_n = \bot$ for some $\mathcal{F}_i \in S$.

□

5.18. Separability of filters

PROPOSITION 576. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.