2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. Let $a \in \mathfrak{A}$. Then the following are equivalent:
(a) a is prime.
(b) For every $A \in \mathfrak{Z}$ exactly one of $\{A, \overline{A}\}$ is in $\operatorname{up} a$.
(c) a is an atom of $\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°.

3°a⇒3°b. Let a be prime. Then $A \sqcup^{\mathfrak{Z}} \overline{A} = \top^{\mathfrak{A}} \in \operatorname{up} a$. Therefore $A \in \operatorname{up} a \vee \overline{A} \in \operatorname{up} a$. But since $A \sqcap^{\mathfrak{Z}} \overline{A} = \bot^{\mathfrak{Z}}$ it is impossible $A \in \operatorname{up} a \wedge \overline{A} \in \operatorname{up} a$.

3°b⇒3°c. Obviously $a \neq \bot^{\mathfrak{A}}$.
Let a filter $b \sqsubset a$. Take $X \in \operatorname{up} b$ such that $X \notin \operatorname{up} a$. Then $\overline{X} \in \operatorname{up} a$ because a is prime and thus $\overline{X} \in \operatorname{up} b$. So $\bot^{\mathfrak{Z}} = X \sqcap^{\mathfrak{Z}} \overline{X} \in \operatorname{up} b$ and thus $b = \bot^{\mathfrak{A}}$. So a is atomic.

3°c⇒3°a. By the previous proposition.

□

5.16. Stars for filters

THEOREM 563. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is an ideal base and has least element.
3°. ∂a is a free star for each $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $\mathfrak{A}$ is a distributive lattice by the corollary 528. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is binarily join-closed by corollary 531. So we can apply the theorem 551.

□

5.16.1. Stars of Filters on Boolean Lattices.

In this section we will consider the set of filters $\mathfrak{A}$ on a boolean lattice $\mathfrak{Z}$.

THEOREM 564. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\partial \mathcal{A} = \neg \langle \neg \rangle^* \operatorname{up} \mathcal{A} = \langle \neg \rangle^* \neg \operatorname{up} \mathcal{A}$ and $\operatorname{up} \mathcal{A} = \neg \langle \neg \rangle^* \partial \mathcal{A} = \langle \neg \rangle^* \neg \partial \mathcal{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. Because of properties of diagram (1), it is enough to prove just $\partial \mathcal{A} = \neg \langle \neg \rangle^* \operatorname{up} \mathcal{A}$. Really, $X \in \operatorname{up} \mathcal{A} \Leftrightarrow X \sqsupseteq \mathcal{A} \Leftrightarrow \overline{X} \asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{X} \notin \partial \mathcal{A}$ for any $X \in \mathfrak{Z}$ (taking into account theorems 532, 534, and lemma 548).

□

COROLLARY 565. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. ∂ is an order isomorphism from $\mathfrak{A}$ to $\mathfrak{S}(\mathfrak{Z})$.

PROOF. By properties of the diagram (1). □

COROLLARY 566. The following is an implications tuple: