Suppose now that a is a coatom of $\mathfrak{A}$. To finish the proof it is enough to show that a is principal. (Then a is non-greatest and thus is a coatom of $\mathfrak{Z}$.)

Suppose a is non-principal. Then obviously exist two distinct elements x and y of the core such that $x, y \in \operatorname{up} a$. Thus a is not an atom of $\mathfrak{A}$.

□

Corollary 558. Coatoms of the set of filters on a set U are exactly sets $U \setminus \{x\}$ where $x \in U$.

Proposition 559. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a coatomic poset.
3°. $\mathfrak{A}$ is coatomic.

Proof.

1°⇒2°. Obvious.

2°⇒2°. Suppose $\mathcal{A} \in \mathfrak{A}$ and $\mathcal{A} \neq \top^{\mathfrak{A}}$. Then there exists $A \in \operatorname{up} \mathcal{A}$ such that A is not greatest element of $\mathfrak{Z}$. Consequently there exists a coatom $a \in \mathfrak{Z}$ such that $a \sqsupseteq A$. Thus $a \in \operatorname{up} \mathcal{A}$ and a is not greatest.

□

5.15. Prime Filtrator Elements

Definition 560. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. *Prime* filtrator elements are such $a \in \mathfrak{A}$ that $\operatorname{up} a$ is a free star (in lattice $\mathfrak{Z}$).

Proposition 561. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is an ideal base.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with binarily join-closed core, where $\mathfrak{A}$ is a starrish join-semilattice and $\mathfrak{Z}$ is a join-semilattice.
4°. Atomic elements of this filtrator are prime.

Proof.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily join-closed core by the theorem 531, $\mathfrak{A}$ is a distributive lattice by theorem 528.

3°⇒4°. Let a be an atom of the lattice $\mathfrak{A}$. We have for every $X, Y \in \mathfrak{Z}$

$$\begin{aligned} X \sqcup^{\mathfrak{Z}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \sqsupseteq a &\Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \not\asymp^{\mathfrak{A}} a &\Leftrightarrow \\ X \not\asymp^{\mathfrak{A}} a \vee Y \not\asymp^{\mathfrak{A}} a &\Leftrightarrow \\ X \sqsupseteq a \vee Y \sqsupseteq a &\Leftrightarrow \\ X \in \operatorname{up} a \vee Y \in \operatorname{up} a&. \end{aligned}$$

□

The following theorem is essentially borrowed from [19]:

Theorem 562. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.