3°. $\mathfrak{A}$ is a co-frame.

The below theorem uses the notation and results from section 3.9.

THEOREM 530. If $\mathfrak{A}$ is a co-frame and L is a bounded distributive lattice which, then $\operatorname{Join}(L, \mathfrak{A})$ is also a co-frame.

PROOF. Let $F = \uparrow \circ \bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \operatorname{Up}(\mathfrak{A})$; F is a co-nucleus by above.

Since $\operatorname{Up}(\mathfrak{A}) \cong \mathbf{Pos}(\mathfrak{A}, 2)$ by proposition 337, we may regard F as a co-nucleus on $\mathbf{Pos}(\mathfrak{A}, 2)$.

$\operatorname{Join}(L, \mathfrak{A}) \cong \operatorname{Join}(L, \operatorname{Fix}(F))$ by corollary 340.

$\operatorname{Join}(L, \operatorname{Fix}(F)) \cong \operatorname{Fix}(\operatorname{Join}(L, F))$ by lemma 348.

By corollary 347 the function $\operatorname{Join}(L, F)$ is a co-nucleus on $\operatorname{Join}(L, \mathbf{Pos}(\mathfrak{A}, 2))$.

$$\begin{aligned} \operatorname{Join}(L, \mathbf{Pos}(\mathfrak{A}, 2)) &\cong \quad \text{(by lemma 350)} \\ \mathbf{Pos}(\mathfrak{A}, \operatorname{Join}(L, 2)) &\cong \\ \mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X)). \end{aligned}$$

$\mathfrak{F}(X)$ is a co-frame by corollary 529. Thus $\mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X))$ is a co-frame by lemma 350.

Thus $\operatorname{Join}(L, \mathfrak{A})$ is isomorphic to a poset of fixed points of a co-nucleus on the co-frame $\mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X))$. By lemma 332 $\operatorname{Join}(L, \mathfrak{A})$ is also a co-frame. □

5.9. Misc filtrator properties

THEOREM 531. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with join-closed core.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. The formula $\forall a, b \in \mathfrak{A} : (\operatorname{up} a \supseteq \operatorname{up} b \Rightarrow a \sqsubseteq b)$ is obvious for primary filtrators.

3°⇒4°. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtered filtrator. Let $S \in \mathscr{P}\mathfrak{Z}$ and $\bigsqcup^{\mathfrak{Z}} S$ be defined. We need to prove $\bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{Z}} S$. That $\bigsqcup^{\mathfrak{Z}} S$ is an upper bound for S is obvious. Let $a \in \mathfrak{A}$ be an upper bound for S. It's enough to prove that $\bigsqcup^{\mathfrak{Z}} S \sqsubseteq a$. Really,

$$c \in \operatorname{up} a \Rightarrow c \sqsupseteq a \Rightarrow \forall x \in S : c \sqsupseteq x \Rightarrow c \sqsupseteq \bigsqcup^{\mathfrak{Z}} S \Rightarrow c \in \operatorname{up} \bigsqcup^{\mathfrak{Z}} S;$$

so $\operatorname{up} a \subseteq \operatorname{up} \bigsqcup^{\mathfrak{Z}} S$ and thus $a \sqsupseteq \bigsqcup^{\mathfrak{Z}} S$ because it is filtered.

□

5.10. Characterization of Binarily Meet-Closed Filtrators

THEOREM 532. The following are equivalent for a filtrator $(\mathfrak{A}, \mathfrak{Z})$ whose core is a meet semilattice such that $\forall a \in \mathfrak{A} : \operatorname{up} a \neq \emptyset$:

1°. The filtrator is with binarily meet-closed core.
2°. $\operatorname{up} a$ is a filter for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Let $X, Y \in \operatorname{up} a$. Then $X \sqcap^{\mathfrak{Z}} Y = X \sqcap^{\mathfrak{A}} Y \sqsupseteq a$. That $\operatorname{up} a$ is an upper set is obvious. So taking into account that $\operatorname{up} a \neq \emptyset$, $\operatorname{up} a$ is a filter.