1°. From the previous theorem.

2°. By duality.

3°. Taking into account the lemma, it is enough to prove that $\bigcup S$ is a free star. $\bigcup S$ is not the complement of empty set because $\bot \notin \bigcup S$. For every $A, B \in \mathfrak{Z}$ we have:

$$A \in \bigcup S \vee B \in \bigcup S \Leftrightarrow \exists P \in S : (A \in P \vee B \in P) \Leftrightarrow \exists P \in S : A \sqcup B \in P \Leftrightarrow A \sqcup B \in \bigcup S.$$

4°. By duality.

□

Corollary 514. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with greatest element $\top$.
3°. $\bigsqcup^{\mathfrak{A}} S$ exists and $\operatorname{up} \bigsqcup^{\mathfrak{A}} S = \bigcap \langle \operatorname{up} \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A} \setminus \{\emptyset\}$.

Proof.

1°⇒2°. Obvious.

2°⇒3°. By the theorem.

□

Corollary 515. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with greatest element $\top$.
3°. $\mathfrak{A}$ is a complete lattice.

We will denote meets and joins on the lattice of filters just as $\sqcap$ and $\sqcup$.

Proposition 516. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over an ideal base.
3°. $\mathfrak{A}$ is a join-semilattice and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$

$$\operatorname{up}(\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{B}) = \operatorname{up} \mathcal{A} \cap \operatorname{up} \mathcal{B}.$$

Proof.

1°⇒2°. Obvious.

2°⇒3°. Taking in account the lemma it is enough to prove that $R = \operatorname{up} \mathcal{A} \cap \operatorname{up} \mathcal{B}$ is a filter.

R is nonempty because we can take $X \in \operatorname{up} \mathcal{A}$ and $Y \in \operatorname{up} \mathcal{B}$ and $Z \sqsupseteq X \wedge Z \sqsupseteq Y$ and then $R \ni Z$.

Let $A, B \in R$. Then $A, B \in \operatorname{up} \mathcal{A}$; so exists $C \in \operatorname{up} \mathcal{A}$ such that $C \sqsubseteq A \wedge C \sqsubseteq B$. Analogously exists $D \in \operatorname{up} \mathcal{B}$ such that $D \sqsubseteq A \wedge D \sqsubseteq B$. Take $E \sqsupseteq C \wedge E \sqsupseteq D$. Then $E \in \operatorname{up} \mathcal{A}$ and $E \in \operatorname{up} \mathcal{B}$; $E \in R$ and $E \sqsubseteq A \wedge E \sqsubseteq B$. So R is a filter base.

That R is an upper set is obvious.

□

Theorem 517. Let $\mathfrak{Z}$ be a distributive lattice. Then

1°. $\bigsqcap^{\mathfrak{F}(\mathfrak{Z})} S = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i=0,\ldots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$;
2°. $\bigsqcup^{\mathfrak{I}(\mathfrak{Z})} S = \left\{ \frac{K_0 \sqcup^{\mathfrak{Z}} \cdots \sqcup^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i=0,\ldots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$.