OBVIOUS 507.

1°. Every up-aligned filtrator is weakly up-aligned.
2°. Every down-aligned filtrator is weakly down-aligned.

OBVIOUS 508.

1°. Every primary filtrator is weakly down-aligned.
2°. Every primary filtrator is weakly up-aligned.

5.8. More advanced properties of filters

5.8.1. Formulas for Meets and Joins of Filters.

LEMMA 509. If f is an order embedding from a poset $\mathfrak{A}$ to a complete lattice $\mathfrak{B}$ and $S \in \mathscr{P}\mathfrak{A}$ and there exists such $\mathcal{F} \in \mathfrak{A}$ that $f\mathcal{F} = \bigsqcup^{\mathfrak{B}} \langle f\rangle^* S$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f\rangle^* S$.

PROOF. f is an order isomorphism from $\mathfrak{A}$ to $\mathfrak{B}|_{\langle f\rangle^*\mathfrak{A}}$. $f\mathcal{F} \in \mathfrak{B}|_{\langle f\rangle^*\mathfrak{A}}$.
Consequently, $\bigsqcup^{\mathfrak{B}} \langle f\rangle^* S \in \mathfrak{B}|_{\langle f\rangle^*\mathfrak{A}}$ and $\bigsqcup^{\mathfrak{B}|_{\langle f\rangle^*\mathfrak{A}}} \langle f\rangle^* S = \bigsqcup^{\mathfrak{B}} \langle f\rangle^* S$.
$f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}|_{\langle f\rangle^*\mathfrak{A}}} \langle f\rangle^* S$ because f is an order isomorphism.
Combining, $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f\rangle^* S$. □

COROLLARY 510. If $\mathfrak{B}$ is a complete lattice and $\mathfrak{A}$ is its subset and $S \in \mathscr{P}\mathfrak{A}$ and $\bigsqcup^{\mathfrak{B}} S \in \mathfrak{A}$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $\bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} S$.

EXERCISE 511. The below theorem does not work for $S = \emptyset$. Formulate the general case.

THEOREM 512.

1°. If $\mathfrak{Z}$ is a meet-semilattice, then $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S$ exists and $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S = \bigcap S$ for every bounded above set $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$.
2°. If $\mathfrak{Z}$ is a join-semilattice, then $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S$ exists and $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S = \bigcap S$ for for every bounded below set $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$.

PROOF.

1°. Taking into account the lemma, it is enough to prove that $\bigcap S$ is a filter. Let's prove that $\bigcap S$ is nonempty. There is an upper bound $\mathcal{T}$ of S. Take arbitrary $T \in \mathcal{T}$. We have $T \in \mathcal{X}$ for every $\mathcal{X} \in S$. Thus S is nonempty.
For every $A, B \in \mathfrak{Z}$ we have:

$$A, B \in \bigcap S \Leftrightarrow \forall P \in S : A, B \in P \Leftrightarrow \forall P \in S : A \sqcap B \in P \Leftrightarrow A \sqcap B \in \bigcap S.$$

So $\bigcap S$ is a filter.
2°. By duality.

□

THEOREM 513.

1°. If $\mathfrak{Z}$ is a meet-semilattice with greatest element, then $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S$ exists and $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S = \bigcap S$ for every $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$.
2°. If $\mathfrak{Z}$ is a join-semilattice with least element, then $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S$ exists and $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S = \bigcap S$ for every $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$.
3°. If $\mathfrak{Z}$ is a join-semilattice with least element, then $\bigsqcup^{\mathfrak{S}(\mathfrak{Z})} S$ exists and $\bigsqcup^{\mathfrak{S}(\mathfrak{Z})} S = \bigcup S$ for every $S \in \mathscr{P}\mathfrak{S}(\mathfrak{Z})$.
4°. If $\mathfrak{Z}$ is a meet-semilattice with greatest element, then $\bigsqcap^{\mathfrak{M}(\mathfrak{Z})} S$ exists and $\bigsqcap^{\mathfrak{M}(\mathfrak{Z})} S = \bigcup S$ for every $S \in \mathscr{P}\mathfrak{M}(\mathfrak{Z})$.

PROOF.