- Free stars are upper sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Rightarrow \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are lower sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Rightarrow \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

Proposition 471. The following are equivalent:

1°. F is a free star.
2°. $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and $F \neq \mathscr{P}\mathfrak{Z}$.
3°. $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Rightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and F is an upper set and $F \neq \mathscr{P}\mathfrak{Z}$.

Proof.

1°⇔2°. The following is a chain of equivalencies:

$$\exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B) \Leftrightarrow A \notin F \wedge B \notin F;$$
$$\forall Z \in \overline{F} : \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B) \Leftrightarrow A \in F \vee B \in F;$$
$$\forall Z \in \mathfrak{Z} : (Z \notin F \Rightarrow \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B)) \Leftrightarrow A \in F \vee B \in F;$$
$$\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F.$$

2°⇒3°. Let $A = B \in F$. Then $A \in F \vee B \in F$. So $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F)$ that is $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \Rightarrow Z \in F)$ that is F is an upper set.

3°⇒2°. We need to prove that F is an upper set. let $A \in F$ and $A \sqsubseteq B \in \mathfrak{Z}$. Then $A \in F \vee B \in F$ and thus $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F)$ that is $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq B \Rightarrow Z \in F)$ and so $B \in F$.

□

Corollary 472. The following are equivalent:

1°. F is a mixer.
2°. $\forall Z \in \mathfrak{Z} : (Z \sqsubseteq A \wedge Z \sqsubseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and $F \neq \mathscr{P}\mathfrak{Z}$.
3°. $\forall Z \in \mathfrak{Z} : (Z \sqsubseteq A \wedge Z \sqsubseteq B \Rightarrow Z \in F) \Rightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and F is an lower set and $F \neq \mathscr{P}\mathfrak{Z}$.

Obvious 473.

1°. A free star cannot contain the least element of the poset.
2°. A mixer cannot contain the greatest element of the poset.

5.6.4. Filters, ideals, free stars, and mixers on semilattices.

Proposition 474.

- Free stars are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg\exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Free stars are upper sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftarrow \neg\exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg\exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are lower sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftarrow \neg\exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

Proof. By duality. □

By duality and and an above theorem about filters, we have:

Proposition 475.