5.6. Alternative primary filtrators

5.6.1. Lemmas.

LEMMA 465. A set F is a lower set iff $\overline{F}$ is an upper set.

PROOF. $X \in \overline{F} \wedge Z \sqsupseteq X \Rightarrow Z \in \overline{F}$ is equivalent to $Z \in F \Rightarrow X \in F \vee Z \not\sqsupseteq X$ is equivalent $Z \in F \Rightarrow (Z \sqsupseteq X \Rightarrow X \in F)$ is equivalent $Z \in F \wedge X \sqsubseteq Z \Rightarrow X \in F$. □

PROPOSITION 466. Let $\mathfrak{Z}$ be a poset with least element $\bot$. Then for upper set F we have $F \neq \mathscr{P}\mathfrak{Z} \Leftrightarrow \bot \notin F$.

PROOF.

$\Rightarrow$. If $\bot \in F$ then $F = \mathscr{P}\mathfrak{Z}$ because F is an upper set.
$\Leftarrow$. Obvious.

□

5.6.2. Informal introduction. We have already defined filters on a poset. Now we will define three other sets which are order-isomorphic to the set of filters on a poset: ideals ($\mathfrak{I}$), free stars ($\mathfrak{S}$), and mixers ($\mathfrak{M}$).

These four kinds of objects are related through commutative diagrams. First we will paint an informal commutative diagram (it makes no formal sense because it is not pointed the poset for which the filters are defined):

$$\begin{array}{ccc} \mathfrak{F} & \xleftrightarrow{\langle \mathrm{dual} \rangle^*} & \mathfrak{I} \\ \updownarrow \neg & & \updownarrow \neg \\ \mathfrak{M} & \xleftrightarrow{\langle \mathrm{dual} \rangle^*} & \mathfrak{S} \end{array}$$

Then we can define ideals, free stars, and mixers as sets following certain formulas. You can check that the intuition behind these formulas follows the above commutative diagram. (That is transforming these formulas by the course of the above diagram, you get formulas of the other objects in this list.)

After this, we will paint some formal commutative diagrams similar to the above diagram but with particular posets at which filters, ideals, free stars, and mixers are defined.

5.6.3. Definitions of ideals, free stars, and mixers. *Filters* and *ideals* are well known concepts. The terms *free stars* and *mixers* are my new terminology.

Recall that *filters* are nonempty sets F with $A, B \in F \Leftrightarrow \exists Z \in F : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 467. *Ideals* are nonempty sets F with $A, B \in F \Leftrightarrow \exists Z \in F : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 468. *Free stars* are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Leftrightarrow \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 469. *Mixers* are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Leftrightarrow \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

By duality and and an above theorem about filters, we have:

PROPOSITION 470.

- Filters are nonempty upper sets F with $A, B \in F \Rightarrow \exists Z \in F : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Ideals are nonempty lower sets F with $A, B \in F \Rightarrow \exists Z \in F : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).