5.5. Filtrators

$(\mathfrak{F}, \mathfrak{P})$ is a poset and its subset (with induced order on the subset). I call pairs of a poset and its subset like this *filtrators*.

Definition 435. I will call a *filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a poset $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$. I call $\mathfrak{A}$ the *base* of the filtrator and $\mathfrak{Z}$ the *core* of the filtrator. I will also say that $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator *over* poset $\mathfrak{Z}$.

I will denote $\operatorname{base}(\mathfrak{A}, \mathfrak{Z}) = \mathfrak{A}$, $\operatorname{core}(\mathfrak{A}, \mathfrak{Z}) = \mathfrak{Z}$ for a filtrator $(\mathfrak{A}, \mathfrak{Z})$.

While *filters* are customary and well known mathematical objects, the concept of *filtrators* is probably first researched by me.

When speaking about filters, we will imply that we consider the filtrator $(\mathfrak{F}, \mathfrak{P})$ or what is the same (as we equate principal filters with base elements) the filtrator $(\mathfrak{F}, \mathfrak{Z})$.

Definition 436. I will call a *lattice filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a lattice $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 437. I will call a *complete lattice filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a complete lattice $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 438. I will call a *central filtrator* a filtrator $(\mathfrak{A}, Z(\mathfrak{A}))$ where $Z(\mathfrak{A})$ is the center of a bounded lattice $\mathfrak{A}$.

Definition 439. I will call *element* of a filtrator an element of its base.

Definition 440. $\operatorname{up}^{\mathfrak{Z}} a = \operatorname{up} a = \left\{ \frac{c \in \mathfrak{Z}}{c \sqsupseteq a} \right\}$ for an element a of a filtrator.

Definition 441. $\operatorname{down}^{\mathfrak{Z}} a = \operatorname{down} a = \left\{ \frac{c \in \mathfrak{Z}}{c \sqsubseteq a} \right\}$ for an element a of a filtrator.

Obvious 442. "up" and "down" are dual.

Our main purpose here is knowing properties of the core of a filtrator to infer properties of the base of the filtrator, specifically properties of $\operatorname{up} a$ for every element a.

Definition 443. I call a filtrator *with join-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\bigsqcup^{\mathfrak{Z}} S = \bigsqcup^{\mathfrak{A}} S$ whenever $\bigsqcup^{\mathfrak{Z}} S$ exists for $S \in \mathscr{P}\mathfrak{Z}$.

Definition 444. I call a filtrator *with meet-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\bigsqcap^{\mathfrak{Z}} S = \bigsqcap^{\mathfrak{A}} S$ whenever $\bigsqcap^{\mathfrak{Z}} S$ exists for $S \in \mathscr{P}\mathfrak{Z}$.

Definition 445. I call a filtrator with *binarily join-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $a \sqcup^{\mathfrak{Z}} b = a \sqcup^{\mathfrak{A}} b$ whenever $a \sqcup^{\mathfrak{Z}} b$ exists for $a, b \in \mathfrak{Z}$.

Definition 446. I call a filtrator with *binarily meet-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $a \sqcap^{\mathfrak{Z}} b = a \sqcap^{\mathfrak{A}} b$ whenever $a \sqcap^{\mathfrak{Z}} b$ exists for $a, b \in \mathfrak{Z}$.

Definition 447. *Prefiltered filtrator* is a filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that "up" is injective.

Definition 448. *Filtered filtrator* is a filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that

$$\forall a, b \in \mathfrak{A} : (\operatorname{up} a \supseteq \operatorname{up} b \Rightarrow a \sqsubseteq b).$$

Theorem 449. A filtrator $(\mathfrak{A}, \mathfrak{Z})$ is filtered iff $\forall a \in \mathfrak{A} : a = \bigsqcap^{\mathfrak{A}} \operatorname{up} a$.

Proof.

$\Leftarrow$. $\operatorname{up} a \supseteq \operatorname{up} b \Rightarrow \bigsqcap^{\mathfrak{A}} \operatorname{up} a \sqsubseteq \bigsqcap^{\mathfrak{A}} \operatorname{up} b \Rightarrow a \sqsubseteq b$.