5.3.3. Order of filters. Principal filters. I will make the set of filters $\mathfrak{F}$ into a poset by the order defined by the formula: $a \sqsubseteq b \Leftrightarrow a \supseteq b$.

DEFINITION 423. The principal filter corresponding to an element $a \in \mathfrak{Z}$ is

$$\uparrow a = \left\{ \frac{x \in \mathfrak{Z}}{x \sqsupseteq a} \right\}.$$

Elements of $\mathfrak{P} = \langle \uparrow \rangle^* \mathfrak{Z}$ are called *principal filters*.

OBVIOUS 424. Principal filters are filters.

OBVIOUS 425. $\uparrow$ is an order embedding from $\mathfrak{Z}$ to $\mathfrak{F}$.

COROLLARY 426. $\uparrow$ is an order isomorphism between $\mathfrak{Z}$ and $\mathfrak{P}$.

We will equate principal filters with corresponding elements of the base poset (in the same way as we equate for example nonnegative whole numbers and natural numbers).

PROPOSITION 427. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$.

PROOF. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow K \subseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$. □

5.4. Filters on a Set

Consider filters on the poset $\mathfrak{Z} = \mathscr{P}\mathfrak{U}$ (where $\mathfrak{U}$ is some fixed set) with the order $A \sqsubseteq B \Leftrightarrow A \subseteq B$ (for $A, B \in \mathscr{P}\mathfrak{A}$).

In fact, it is a complete atomistic boolean lattice with $\bigsqcap S = \bigcap S$, $\bigsqcup S = \bigcup S$, $\overline{A} = \mathfrak{U} \setminus A$ for every $S \in \mathscr{P}\mathscr{P}\mathfrak{U}$ and $A \in \mathscr{P}\mathfrak{U}$, atoms being one-element sets.

DEFINITION 428. I will call a filter on the lattice of all subsets of a given set $\mathfrak{U}$ as a *filter on set*.

DEFINITION 429. I will denote the set on which a filter $\mathcal{F}$ is defined as $\operatorname{Base}(\mathcal{F})$.

OBVIOUS 430. $\operatorname{Base}(\mathcal{F}) = \bigcup \mathcal{F}$.

PROPOSITION 431. The following are equivalent for a non-empty set $F \in \mathscr{P}\mathscr{P}\mathfrak{U}$:

1°. F is a filter.
2°. $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set.
3°. $\forall X, Y \in \mathscr{P}\mathfrak{U} : (X, Y \in F \Leftrightarrow X \cap Y \in F)$.

PROOF. By theorem 420. □

OBVIOUS 432. The minimal filter on $\mathscr{P}\mathfrak{U}$ is $\mathscr{P}\mathfrak{U}$.

OBVIOUS 433. The maximal filter on $\mathscr{P}\mathfrak{U}$ is $\{\mathfrak{U}\}$.

I will denote $\uparrow A = \uparrow^{\mathfrak{U}} A = \uparrow^{\mathscr{P}\mathfrak{U}} A$. (The distinction between conflicting notations $\uparrow^{\mathfrak{U}} A$ and $\uparrow^{\mathscr{P}\mathfrak{U}} A$ will be clear from the context.)

PROPOSITION 434. Every filter on a finite set is principal.

PROOF. Let $\mathcal{F}$ be a filter on a finite set. Then obviously $\mathcal{F} = \bigsqcap^{\mathfrak{Z}} \operatorname{up} \mathcal{F}$ and thus $\mathcal{F}$ is principal. □