consists of all intervals $]-\epsilon;\epsilon[$ for all $\epsilon > 0$ and also all subsets of $\mathbb{R}$ containing such intervals as subsets. Informally speaking, this is the greatest filter contained in every interval $]-\epsilon;\epsilon[$ for all $\epsilon > 0$.

DEFINITION 402. A filter on a set $\mho$ is a $\mathcal{F} \in \mathscr{P}\mathscr{P}\mho$ such that:

1°. $\forall A, B \in \mathcal{F} : A \cap B \in \mathcal{F}$;
2°. $\forall A, B \in \mathscr{P}\mho : (A \in \mathcal{F} \wedge B \supseteq A \Rightarrow B \in \mathcal{F})$.

EXERCISE 403. Verify that the above introduced infinitely small interval near 0 on the real line is a filter on $\mathbb{R}$.

EXERCISE 404. Describe "the neighborhood of positive infinity" filter on $\mathbb{R}$.

DEFINITION 405. A filter not containing empty set is called a *proper filter*.

OBVIOUS 406. The non-proper filter is $\mathscr{P}\mho$.

REMARK 407. Some other authors require that all filters are proper. This is a stupid idea and we allow non-proper filters, in the same way as we allow to use the number 0.

5.2.2. Intro to filters on a meet-semilattice. A trivial generalization of the above:

DEFINITION 408. A filter on a meet-semilattice $\mathfrak{Z}$ is a $\mathcal{F} \in \mathscr{P}\mathfrak{Z}$ such that:

1°. $\forall A, B \in \mathcal{F} : A \sqcap B \in \mathcal{F}$;
2°. $\forall A, B \in \mathfrak{Z} : (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

5.2.3. Intro to filters on a poset.

DEFINITION 409. A filter on a poset $\mathfrak{Z}$ is a $\mathcal{F} \in \mathscr{P}\mathfrak{Z}$ such that:

1°. $\forall A, B \in \mathcal{F} \exists C \in \mathcal{F} : C \sqsubseteq A, B$;
2°. $\forall A, B \in \mathfrak{Z} : (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

It is easy to show (and there is a proof of it somewhere below) that this coincides with the above definition in the case if $\mathfrak{Z}$ is a meet-semilattice.

5.3. Filters on a poset

5.3.1. Filters on posets. Let $\mathfrak{Z}$ be a poset.

DEFINITION 410. *Filter base* is a nonempty subset F of $\mathfrak{Z}$ such that

$$\forall X, Y \in F \exists Z \in F : (Z \sqsubseteq X \wedge Z \sqsubseteq Y).$$

DEFINITION 411. *Ideal base* is a nonempty subset F of $\mathfrak{Z}$ such that

$$\forall X, Y \in F \exists Z \in F : (Z \sqsupseteq X \wedge Z \sqsupseteq Y).$$

OBVIOUS 412. Ideal base is the dual of filter base.

OBVIOUS 413.

1°. A poset with a lowest element is a filter base.
2°. A poset with a greatest element is an ideal base.

OBVIOUS 414.

1°. A meet-semilattice is a filter base.
2°. A join-semilattice is an ideal base.

OBVIOUS 415. A nonempty chain is a filter base and an ideal base.

DEFINITION 416. *Filter* is a subset of $\mathfrak{Z}$ which is both a filter base and an upper set.