4°. From two previous formulas.
5°. $X\ [f]^*\ Y \Leftrightarrow \exists \alpha \in X, \beta \in Y : \alpha\ f\ \beta \Leftrightarrow \exists \alpha \in X, \beta \in Y : \{\alpha\}\ [f]^*\ \{\beta\}$.
6°. Obvious.

□

COROLLARY 377. For a **Rel**-morphism f we have:

1°. $\langle f\rangle^* \bigsqcup S = \bigsqcup \langle\langle f\rangle^*\rangle^* S$ for $S \in \mathscr{P}\mathscr{T}\operatorname{Src} f$;
2°. $\bigsqcup S\ [f]^*\ Y \Leftrightarrow \exists X \in S : X\ [f]^*\ Y$ for $S \in \mathscr{P}\mathscr{T}\operatorname{Src} f$;
3°. $X\ [f]^* \bigsqcup T \Leftrightarrow \exists Y \in T : X\ [f]^*\ Y$ for $T \in \mathscr{P}\mathscr{T}\operatorname{Dst} f$;
4°. $\bigsqcup S\ [f]^* \bigsqcup T \Leftrightarrow \exists X \in S, Y \in T : X\ [f]^*\ Y$ for $S \in \mathscr{P}\mathscr{T}\operatorname{Src} f$, $T \in \mathscr{P}\mathscr{T}\operatorname{Dst} f$;
5°. $X\ [f]^*\ Y \Leftrightarrow \exists x \in \operatorname{atoms} X, y \in \operatorname{atoms} Y : x\ [f]^*\ y$ for $X \in \mathscr{T}\operatorname{Src} f$, $Y \in \mathscr{T}\operatorname{Dst} f$;
6°. $\langle f\rangle^* X = \bigsqcup \langle\langle f\rangle^*\rangle^* \operatorname{atoms} X$ for $X \in \mathscr{T}\operatorname{Src} f$.

COROLLARY 378. A **Rel**-morphism f can be restored knowing either $\langle f\rangle^* x$ for atoms $x \in \mathscr{T}\operatorname{Src} f$ or $x\ [f]^*\ y$ for atoms $x \in \mathscr{T}\operatorname{Src} f$, $y \in \mathscr{T}\operatorname{Dst} f$.

PROPOSITION 379. Let A, B be sets, R be a set of binary relations.

1°. $\langle \bigcup R\rangle^* X = \bigcup_{f\in R} \langle f\rangle^* X$ for every set X;
2°. $\langle \bigcap R\rangle^* \{\alpha\} = \bigcap_{f\in R} \langle f\rangle^* \{\alpha\}$ for every α, if R is nonempty;
3°. $X\ [\bigcup R]^*\ Y \Leftrightarrow \exists f \in R : X\ [f]^*\ Y$ for every sets X, Y;
4°. $\alpha\ (\bigcap R)\ \beta \Leftrightarrow \forall f \in R : \alpha\ f\ \beta$ for every α and β, if R is nonempty.

PROOF.

1°.

$$y \in \left\langle \bigcup R \right\rangle^* X \Leftrightarrow \exists x \in X : x \left(\bigcup R\right) y \Leftrightarrow \exists x \in X, f \in R : x\ f\ y \Leftrightarrow \exists f \in R : y \in \langle f\rangle^* X \Leftrightarrow y \in \bigcup_{f\in R} \langle f\rangle^* X.$$

2°.

$$y \in \left\langle \bigcap R \right\rangle^* \{\alpha\} \Leftrightarrow \forall f \in R : \alpha\ f\ y \Leftrightarrow \forall f \in R : y \in \langle f\rangle^* \{\alpha\} \Leftrightarrow y \in \bigcap_{f\in R} \langle f\rangle^* \{\alpha\}.$$

3°.

$$X \left[\bigcup R\right]^* Y \Leftrightarrow \exists x \in X, y \in Y : x \left(\bigcup R\right) y \Leftrightarrow \exists x \in X, y \in Y, f \in R : x\ f\ y \Leftrightarrow \exists f \in R : X\ [f]^*\ Y.$$

4°. Obvious.

□

COROLLARY 380. Let A, B be sets, $R \in \mathscr{P}\mathbf{Rel}(A, B)$.

1°. $\langle \bigsqcup R\rangle^* X = \bigsqcup_{f\in R} \langle f\rangle^* X$ for $X \in \mathscr{T}A$;
2°. $\langle \bigsqcap R\rangle^* x = \bigsqcap_{f\in R} \langle f\rangle^* x$ for atomic $x \in \mathscr{T}A$;
3°. $X\ [\bigsqcup R]^*\ Y \Leftrightarrow \exists f \in R : X\ [f]^*\ Y$ for $X \in \mathscr{T}A$, $Y \in \mathscr{T}B$;
4°. $x\ [\bigsqcap R]^*\ y \Leftrightarrow \forall f \in R : x\ [f]^*\ y$ for every atomic $x \in \mathscr{T}A$, $y \in \mathscr{T}B$.

PROPOSITION 381. $X\ [g \circ f]^*\ Z \Leftrightarrow \exists \beta : (X\ [f]^*\ \{\beta\} \wedge \{\beta\}\ [g]^*\ Z)$ for every binary relation f and sets X and Y.