OBVIOUS 372. $\left[f^{-1}\right]^*=[f]^{*-1}$ for every **Rel**-morphism f.

OBVIOUS 373. $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ for every composable **Rel**-morphisms f and g.

PROPOSITION 374. $\langle g\circ f\rangle^*=\langle g\rangle^*\circ\langle f\rangle^*$ for every composable **Rel**-morphisms f and g.

PROOF. Exercise. □

PROPOSITION 375. The above definitions of monovalued morphisms of **Rel** and of injective morphisms of **Set** coincide with how mathematicians usually define monovalued functions (that is morphisms of **Set**) and injective functions.

PROOF. Let f be a **Rel**-morphism $A\to B$.
The following are equivalent:

- f is a monovalued relation;
- $\forall x\in A,y_0,y_1\in B:(x\ f\ y_0\wedge x\ f\ y_1\Rightarrow y_0=y_1)$;
- $\forall x\in A,y_0,y_1\in B:(y_0\neq y_1\Rightarrow\neg(x\ f\ y_0)\vee\neg(x\ f\ y_1))$;
- $\forall y_0,y_1\in B\forall x\in A:(y_0\neq y_1\Rightarrow\neg(x\ f\ y_0)\vee\neg(x\ f\ y_1))$;
- $\forall y_0,y_1\in B:(y_0\neq y_1\Rightarrow\forall x\in A:(\neg(x\ f\ y_0)\vee\neg(x\ f\ y_1)))$;
- $\forall y_0,y_1\in B:(\exists x\in A:(x\ f\ y_0\wedge x\ f\ y_1)\Rightarrow y_0=y_1)$;
- $\forall y_0,y_1\in B:y_0\ (f\circ f^{-1})\ y_1\Rightarrow y_0=y_1$;
- $f\circ f^{-1}\sqsubseteq 1_B$.

Let now f be a **Set**-morphism $A\to B$.
The following are equivalent:

- f is an injective function;
- $\forall y\in B,a,b\in A:(a\ f\ y\wedge b\ f\ y\Rightarrow a=b)$;
- $\forall y\in B,a,b\in A:(a\neq b\Rightarrow\neg(a\ f\ y)\vee\neg(b\ f\ y))$;
- $\forall y\in B:(a\neq b\Rightarrow\forall a,b\in A:(\neg(a\ f\ y)\vee\neg(b\ f\ y)))$;
- $\forall y\in B:(\exists a,b\in A:(a\ f\ y\wedge b\ f\ y)\Rightarrow a=b)$;
- $f^{-1}\circ f\sqsubseteq 1_A$.

□

PROPOSITION 376. For a binary relation f we have:

1°. $\langle f\rangle^*\bigcup S=\bigcup\langle\langle f\rangle^*\rangle^*S$ for a set of sets S;
2°. $\bigcup S\ [f]^*\ Y\Leftrightarrow\exists X\in S:X\ [f]^*\ Y$ for a set of sets S;
3°. $X\ [f]^*\bigcup T\Leftrightarrow\exists Y\in T:X\ [f]^*\ Y$ for a set of sets T;
4°. $\bigcup S\ [f]^*\bigcup T\Leftrightarrow\exists X\in S,Y\in T:X\ [f]^*\ Y$ for sets of sets S and T;
5°. $X\ [f]^*\ Y\Leftrightarrow\exists\alpha\in X,\beta\in Y:\{\alpha\}\ [f]^*\ \{\beta\}$ for sets X and Y;
6°. $\langle f\rangle^*X=\bigcup\langle\langle f\rangle^*\rangle^*\operatorname{atoms}X$ for a set X (where $\operatorname{atoms}X=\left\{\frac{\{x\}}{x\in X}\right\}$).

PROOF.

1°.

$$y\in\langle f\rangle^*\bigcup S\Leftrightarrow\exists x\in\bigcup S:x\ f\ y\Leftrightarrow\exists P\in S,x\in P:x\ f\ y\Leftrightarrow\\\exists P\in S:y\in\langle f\rangle^*P\Leftrightarrow\exists Q\in\langle\langle f\rangle^*\rangle^*S:y\in Q\Leftrightarrow y\in\bigcup\langle\langle f\rangle^*\rangle^*S.$$

2°.

$$\bigcup S\ [f]^*\ Y\Leftrightarrow\exists x\in\bigcup S,y\in Y:x\ f\ y\Leftrightarrow\\\exists X\in S,x\in X,y\in Y:x\ f\ y\Leftrightarrow\exists X\in S:X\ [f]^*\ Y.$$

3°. By symmetry.