Proof. $a \in \operatorname{Fix}(\operatorname{Join}(P, F)) \Leftrightarrow a \in F^P \wedge F \circ a = a \Leftrightarrow a \in F^P \wedge \forall x \in P : F(a(x)) = a(x)$.

$a \in \operatorname{Join}(P, \operatorname{Fix}(F)) \Leftrightarrow a \in \operatorname{Fix}(F)^P \Leftrightarrow a \in F^P \wedge \forall x \in P : F(a(x)) = a(x)$.

Thus $\operatorname{Fix}(\operatorname{Join}(P, F)) = \operatorname{Join}(P, \operatorname{Fix}(F))$. That the order of the left and right sides of the equality agrees is obvious. □

Definition 349. $\mathbf{Pos}(\mathfrak{A}, \mathfrak{B})$ is the pointwise ordered poset of monotone maps from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

Lemma 350. If Q, R are JSWLEs and P is a poset, then $\mathbf{Pos}(P, R)$ is a JSWLE and $\mathbf{Pos}(P, \operatorname{Join}(Q, R))$ is isomorphic to $\operatorname{Join}(Q, \mathbf{Pos}(P, R))$. If R is a co-frame, then also $\mathbf{Pos}(P, R)$ is a co-frame.

Proof. Let $f, g \in \mathbf{Pos}(P, R)$. Then $\lambda x \in P : (fx \sqcup gx)$ is obviously monotone and then it is evident that $f \sqcup^{\mathbf{Pos}(P,R)} g = \lambda x \in P : (fx \sqcup gx)$. $\lambda x \in P : \bot^R$ is also obviously monotone and it is evident that $\bot^{\mathbf{Pos}(P,R)} = \lambda x \in P : \bot^R$.

Obviously both $\mathbf{Pos}(P, \operatorname{Join}(Q, R))$ and $\operatorname{Join}(Q, \mathbf{Pos}(P, R))$ are sets of order preserving maps.

Let f be a monotone map.

$f \in \mathbf{Pos}(P, \operatorname{Join}(Q, R))$ iff $f \in \operatorname{Join}(Q, R)^P$ iff $f \in \left\{ \frac{g \in R^Q}{g \text{ preserves finite joins}} \right\}^P$ iff $f \in (R^Q)^P$ and every $g = f(x)$ (for $x \in P$) preserving finite joins. This is bijectively equivalent ($f \mapsto f'$) to $f' \in (R^P)^Q$ preserving finite joins.

$f' \in \operatorname{Join}(Q, \mathbf{Pos}(P, R))$ iff f' preserves finite joins and $f' \in \mathbf{Pos}(P, R)^Q$ iff f' preserves finite joins and $f' \in \left\{ \frac{g \in (R^P)^Q}{g(x) \text{ is monotone}} \right\}$ iff f' preserves finite joins and $f' \in (R^P)^Q$.

So we have proved that $f \mapsto f'$ is a bijection between $\mathbf{Pos}(P, \operatorname{Join}(Q, R))$ and $\operatorname{Join}(Q, \mathbf{Pos}(P, R))$. That it preserves order is obvious.

It remains to prove that if R is a co-frame, then also $\mathbf{Pos}(P, R)$ is a co-frame.

First, we need to prove that $\mathbf{Pos}(P, R)$ is a complete lattice. But it is easy to prove that for every set $S \in \mathscr{P}\mathbf{Pos}(P, R)$ we have $\lambda x \in P : \bigsqcup_{f \in S} f(x)$ and $\lambda x \in P : \bigsqcap_{f \in S} f(x)$ are monotone and thus are the joins and meets on $\mathbf{Pos}(P, R)$.

Next we need to prove that

$$b \sqcup^{\mathbf{Pos}(P,R)} \bigsqcap^{\mathbf{Pos}(P,R)} S = \bigsqcap^{\mathbf{Pos}(P,R)} \left\langle b \sqcup^{\mathbf{Pos}(P,R)} \right\rangle^* S.$$

Really (for every $x \in P$),

$$\left(b \sqcup^{\mathbf{Pos}(P,R)} \bigsqcap^{\mathbf{Pos}(P,R)} S \right) x = b(x) \sqcup \left(\bigsqcap^{\mathbf{Pos}(P,R)} S \right) x = b(x) \sqcup \bigsqcap_{f \in S} f(x) = \bigsqcap_{f \in S} (b(x) \sqcup f(x)) = \bigsqcap_{f \in S} \left(b \sqcup^{\mathbf{Pos}(P,R)} f \right) x = \left(\bigsqcap_{f \in S}^{\mathbf{Pos}(P,R)} \left(b \sqcup^{\mathbf{Pos}(P,R)} f \right) \right) x.$$

Thus $b \sqcup^{\mathbf{Pos}(P,R)} \bigsqcap^{\mathbf{Pos}(P,R)} S = \bigsqcap_{f \in S}^{\mathbf{Pos}(P,R)} \left(b \sqcup^{\mathbf{Pos}(P,R)} f \right) = \bigsqcap^{\mathbf{Pos}(P,R)} \left\langle b \sqcup^{\mathbf{Pos}(P,R)} \right\rangle^* S$. □

Definition 351. $P \cong Q$ means that posets P and Q are isomorphic.