Finally, we have $a \sqsupseteq a'$ if and only if $\uparrow a \subseteq \uparrow a'$, so that $\uparrow : \mathfrak{A} \to \operatorname{Up}(\mathfrak{A})$ maps $\mathfrak{A}$ isomorphically onto its image $\langle\uparrow\rangle^*\mathfrak{A}$. This image is $\operatorname{Fix}(F)$ because if D is any fixed point (i.e. if $D = \uparrow \bigsqcap D$), then D clearly belongs to $\langle\uparrow\rangle^*\mathfrak{A}$; and conversely $\uparrow a$ is always a fixed point of $F = \uparrow \circ \bigsqcap$ since $F(\uparrow a) = \uparrow \bigsqcap \uparrow a = \uparrow a$. □

DEFINITION 341. If $\mathfrak{A}$, $\mathfrak{B}$ are two JSWLEs, then $\operatorname{Join}(\mathfrak{A}, \mathfrak{B})$ is the (ordered pointwise) set of finite joins preserving maps $\mathfrak{A} \to \mathfrak{B}$.

OBVIOUS 342. $\operatorname{Join}(\mathfrak{A}, \mathfrak{B})$ is a JSWLE, where $f \sqcup g$ is given by the formula $(f \sqcup g)(p) = f(p) \sqcup g(p)$, $\bot^{\operatorname{Join}(\mathfrak{A},\mathfrak{B})}$ is given by the formula $\bot^{\operatorname{Join}(\mathfrak{A},\mathfrak{B})}(p) = \bot^{\mathfrak{B}}$.

DEFINITION 343. Let $h : Q \to R$ be a finite joins preserving map. Then by definition $\operatorname{Join}(P, h) : \operatorname{Join}(P, Q) \to \operatorname{Join}(P, R)$ takes $f \in \operatorname{Join}(P, Q)$ into the composition $h \circ f \in \operatorname{Join}(P, R)$.

LEMMA 344. Above defined $\operatorname{Join}(P, h)$ is a finite joins preserving map.

PROOF.

$$(h \circ (f \sqcup f'))x = h(f \sqcup f')x = h(fx \sqcup f'x) = \\ hfx \sqcup hf'x = (h \circ f)x \sqcup (h \circ f')x = ((h \circ f) \sqcup (h \circ f'))x.$$

Thus $h \circ (f \sqcup f') = (h \circ f) \sqcup (h \circ f')$.

$(h \circ \bot^{\operatorname{Join}(P,Q)})x = h\bot^{\operatorname{Join}(P,Q)}x = h\bot^Q = \bot^R$. □

PROPOSITION 345. If $h, h' : Q \to R$ are finite join preserving maps and $h \sqsupseteq h'$, then $\operatorname{Join}(P, h) \sqsupseteq \operatorname{Join}(P, h')$.

PROOF. $\operatorname{Join}(P, h)(f)(x) = (h \circ f)(x) = hfx \sqsupseteq h'fx = (h' \circ f)(x) = \operatorname{Join}(P, h')(f)(x)$. □

LEMMA 346. If $g : Q \to R$ and $h : R \to S$ are finite joins preserving, then the composition $\operatorname{Join}(P, h) \circ \operatorname{Join}(P, g)$ is equal to $\operatorname{Join}(P, h \circ g)$. Also $\operatorname{Join}(P, \mathrm{id}_Q)$ for identity map id_Q on Q is the identity map $\mathrm{id}_{\operatorname{Join}(P,Q)}$ on $\operatorname{Join}(P, Q)$.

PROOF. $\operatorname{Join}(P, h) \operatorname{Join}(P, g)f = \operatorname{Join}(P, h)(g \circ f) = h \circ g \circ f = \operatorname{Join}(P, h \circ g)f$.

$\operatorname{Join}(P, \mathrm{id}_Q)f = \mathrm{id}_Q \circ f = f$. □

COROLLARY 347. If Q is a JSWLE and $F : Q \to Q$ is a co-nucleus, then for any JSWLE P we have that

$$\operatorname{Join}(P, F) : \operatorname{Join}(P, Q) \to \operatorname{Join}(P, Q)$$

is also a co-nucleus.

PROOF. From $\mathrm{id}_Q \sqsupseteq F$ (co-nucleus axiom 1°) we have $\operatorname{Join}(P, \mathrm{id}_Q) \sqsupseteq \operatorname{Join}(P, F)$ and since by the last lemma the left side is the identity on $\operatorname{Join}(P, Q)$, we see that $\operatorname{Join}(P, F)$ also satisfies co-nucleus axiom 1°.

$\operatorname{Join}(P, F) \circ \operatorname{Join}(P, F) = \operatorname{Join}(P, F \circ F)$ by the same lemma and thus $\operatorname{Join}(P, F) \circ \operatorname{Join}(P, F) = \operatorname{Join}(P, F)$ by the second co-nucleus axiom for F, showing that $\operatorname{Join}(P, F)$ satisfies the second co-nucleus axiom.

By an other lemma, we have that $\operatorname{Join}(P, F)$ preserves binary joins, given that F preserves binary joins, which is the third co-nucleus axiom. □

LEMMA 348. $\operatorname{Fix}(\operatorname{Join}(P, F)) = \operatorname{Join}(P, \operatorname{Fix}(F))$ for every JSWLEs P, Q and a join preserving function $F : Q \to Q$.