LEMMA 338. For a complete lattice $\mathfrak{A}$, the map $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves arbitrary meets.

PROOF. Let $S \in \mathscr{P}\operatorname{Up}(\mathfrak{A})$. We have $\bigsqcap S \in \operatorname{Up}(\mathfrak{A})$.
$\bigsqcap \bigsqcap S = \bigsqcap \bigsqcap_{X \in S} X = \bigsqcap_{X \in S} \bigsqcap X$ is what we needed to prove. ◻

LEMMA 339. A complete lattice $\mathfrak{A}$ is a co-frame iff $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins.

PROOF.

⇒. Let $\mathfrak{A}$ be a co-frame. Let $D, D' \in \operatorname{Up}(\mathfrak{A})$. Obviously $\bigsqcap(D \sqcup D') \sqsupseteq \bigsqcap D$ and $\bigsqcap(D \sqcup D') \sqsupseteq \bigsqcap D'$, so $\bigsqcap(D \sqcup D') \sqsupseteq \bigsqcap D \sqcup \bigsqcap D'$.

Also

$$\bigsqcap D \sqcup \bigsqcap D' = \bigcup D \sqcup \bigcup D' = \text{(because } \mathfrak{A} \text{ is a co-frame)} = \bigcup \left\{ \frac{d \sqcup d'}{d \in D, d' \in D'} \right\}.$$

Obviously $d \sqcup d' \in D \cap D'$, thus $\bigsqcap D \sqcup \bigsqcap D' \subseteq \bigcup(D \cap D') = \bigsqcap(D \cap D')$ that is $\bigsqcap D \sqcup \bigsqcap D' \sqsupseteq \bigsqcap(D \cap D')$. So $\bigsqcap(D \sqcup D') = \bigsqcap D \sqcup \bigsqcap D'$ that is $\bigsqcap : \operatorname{Up}(A) \to A$ preserves binary joins.

It preserves nullary joins since $\bigsqcap^{\operatorname{Up}(\mathfrak{A})} \bot^{\operatorname{Up}(\mathfrak{A})} = \bigsqcap^{\operatorname{Up}(\mathfrak{A})} \mathfrak{A} = \bot^{\mathfrak{A}}$.

⇐. Suppose $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins. Let $b \in \mathfrak{A}$, $S \in \mathscr{P}\mathfrak{A}$. Let D be the smallest upper set containing S (so $D = \bigcup \langle\uparrow\rangle^* S$). Then

$$\begin{aligned}
& b \sqcup \bigsqcap S = \\
& \bigsqcap \uparrow b \sqcup \bigcup \bigsqcap \langle\uparrow\rangle^* S = \\
& \bigsqcap \uparrow b \sqcup \bigsqcap \bigcup \langle\uparrow\rangle^* S = \text{(since } \textstyle\bigsqcap \text{ preserves finite joins)} \\
& \bigsqcap\left(\uparrow b \sqcup \bigcup \langle\uparrow\rangle^* S\right) = \\
& \bigcup\left(\uparrow b \cap \bigcup \langle\uparrow\rangle^* S\right) = \\
& \bigsqcap \bigcup_{a \in S} (\uparrow b \cap \uparrow a) = \\
& \bigsqcap \bigcup_{a \in S} \uparrow (b \sqcup a) = \text{(since } \textstyle\bigsqcap \text{ preserves all meets)} \\
& \bigcup_{a \in S} \bigsqcap \uparrow (b \sqcup a) = \\
& \bigcup_{a \in S} (b \sqcup a) = \\
& \bigsqcap_{a \in S} (b \sqcup a).
\end{aligned}$$

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COROLLARY 340. If $\mathfrak{A}$ is a co-frame, then the composition $F = \uparrow \circ \bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \operatorname{Up}(\mathfrak{A})$ is a co-nucleus. The embedding $\uparrow : \mathfrak{A} \to \operatorname{Up}(\mathfrak{A})$ is an isomorphism of $\mathfrak{A}$ onto the co-frame $\operatorname{Fix}(F)$.

PROOF. $D \sqsupseteq F(D)$ follows from theorem 336.

We have $F(F(D)) = F(D)$ for all $D \in \operatorname{Up}(\mathfrak{A})$ since $F(F(D)) = \uparrow \bigsqcap \uparrow \bigsqcap D =$ (because $\bigsqcap \uparrow s = s$ for any s) $= \uparrow \bigsqcap D = F(D)$.

And since both $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ and $\uparrow$ preserve finite joins, F preserves finite joins. Thus F is a co-nucleus.