LEMMA 335. The set $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary meets and joins.

PROOF. Let $S \in \mathscr{P} \operatorname{Up}(\mathfrak{A})$.

Let $X \in \bigcup S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then there is $P \in S$ such that $X \in P$ and thus $Y \in P$ and so $Y \in \bigcup S$. So $\bigcup S \in \operatorname{Up}(\mathfrak{A})$.

Let now $X \in \bigcap S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then $\forall T \in S : X \in T$ and so $\forall T \in S : Y \in T$, thus $Y \in \bigcap S$. So $\bigcap S \in \operatorname{Up}(\mathfrak{A})$. □

THEOREM 336. A poset $\mathfrak{A}$ is a complete lattice iff there is a antitone map $s : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ such that

1°. $s(\uparrow p) = p$ for every $p \in \mathfrak{A}$;

2°. $D \subseteq \uparrow s(D)$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

Moreover, in this case $s(D) = \bigsqcap D$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

PROOF.

$\Rightarrow$. Take $s(D) = \bigsqcap D$.

$\Leftarrow$. $\forall x \in D : x \sqsupseteq s(D)$ from the second formula.

Let $\forall x \in D : y \sqsubseteq x$. Then $x \in \uparrow y$, $D \subseteq \uparrow y$; because s is an antitone map, thus follows $s(D) \sqsupseteq s(\uparrow y) = y$. So $\forall x \in D : y \sqsubseteq s(D)$.

That s is the meet follows from the definition of meets.

It remains to prove that $\mathfrak{A}$ is a complete lattice.

Take any subset S of $\mathfrak{A}$. Let D be the smallest upper set containing S. (It exists because $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary joins.) This is

$$D = \left\{ \frac{x \in \mathfrak{A}}{\exists s \in S : x \sqsupseteq s} \right\}.$$

Any lower bound of D is clearly a lower bound of S since $D \supseteq S$. Conversely any lower bound of S is a lower bound of D. Thus S and D have the same set of lower bounds, hence have the same greatest lower bound. □

PROPOSITION 337. For any poset $\mathfrak{A}$ the following are mutually reverse order isomorphisms between upper sets F (ordered reverse to set-theoretic inclusion) on $\mathfrak{A}$ and order homomorphisms $\varphi : \mathfrak{A}^{\mathrm{op}} \to 2$ (here 2 is the partially ordered set of two elements: 0 and 1 where $0 \sqsubseteq 1$), defined by the formulas

1°. $\varphi(a) = \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$ for every $a \in \mathfrak{A}$;

2°. $F = \varphi^{-1}(1)$.

PROOF. Let $X \in \varphi^{-1}(1)$ and $Y \sqsupseteq X$. Then $\varphi(X) = 1$ and thus $\varphi(Y) = 1$. Thus $\varphi^{-1}(1)$ is a upper set.

It is easy to show that φ defined by the formula 1° is an order homomorphism $\mathfrak{A}^{\mathrm{op}} \to 2$ whenever F is a upper set.

Finally we need to prove that they are mutually inverse. Really: Let φ be defined by the formula 1°. Then take $F' = \varphi^{-1}(1)$ and define $\varphi'(a)$ by the formula 1°. We have

$$\varphi'(a) = \begin{cases} 1 & \text{if } a \in \varphi^{-1}(1) \\ 0 & \text{if } a \notin \varphi^{-1}(1) \end{cases} = \begin{cases} 1 & \text{if } \varphi(a) = 1 \\ 0 & \text{if } \varphi(a) \neq 1 \end{cases} = \varphi(a).$$

Let now F be defined by the formula 2°. Then take $\varphi'(a) = \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$ as defined by the formula 1° and define $F' = \varphi'^{-1}(1)$. Then

$$F' = \varphi'^{-1}(1) = F.$$

□