Currying and uncurrying with a dependent variable. Let X, Z be sets and Y be a function with the domain X. (Vaguely saying, Y is a variable dependent on X.)

The disjoint union $\coprod Y = \bigcup_{i\in\operatorname{dom}Y}(\{i\}\times Y_i) = \left\{\frac{(i,x)}{i\in\operatorname{dom}Y, x\in Y_i}\right\}$.

We will consider variables $x\in X$ and $y\in Y_x$.

Let a function $f\in Z^{\coprod_{i\in X}Y_i}$ (or equivalently $f\in Z^{\coprod Y}$). Then $\operatorname{curry}(f)\in\prod_{i\in X}Z^{Y_i}$ is the function defined by the formula $(\operatorname{curry}(f)x)y = f(x,y)$.

Let now $f\in\prod_{i\in X}Z^{Y_i}$. Then $\operatorname{uncurry}(f)\in Z^{\coprod_{i\in X}Y_i}$ is the function defined by the formula $\operatorname{uncurry}(f)(x,y) = (fx)y$.

OBVIOUS 285.

1°. $\operatorname{uncurry}(\operatorname{curry}(f)) = f$ for every $f\in Z^{\coprod_{i\in X}Y_i}$.
2°. $\operatorname{curry}(\operatorname{uncurry}(f)) = f$ for every $f\in\prod_{i\in X}Z^{Y_i}$.

3.7.2.2. *Functions with ordinal numbers of arguments.* Let Ord be the set of small ordinal numbers.

If X and Y are sets and n is an ordinal number, the set of functions taking n arguments on the set X and returning a value in Y is Y^{X^n}.

The set of all small functions taking ordinal numbers of arguments is $Y^{\bigcup_{n\in\mathrm{Ord}}X^n}$.

I will denote $\operatorname{OrdVar}(X) = \mho^{\bigcup_{n\in\mathrm{Ord}}X^n}$ and call it *ordinal variadic*. ("Var" in this notation is taken from the word *variadic* in the collocation *variadic function* used in computer science.)

3.7.3. On sums of ordinals. Let a be an ordinal-indexed family of ordinals.

PROPOSITION 286. $\coprod a$ with lexicographic order is a well-ordered set.

PROOF. Let S be non-empty subset of $\coprod a$.

Take $i_0 = \min\operatorname{Pr}_0 S$ and $x_0 = \min\left\{\frac{\operatorname{Pr}_1 y}{y\in S, y(0)=i_0}\right\}$ (these exist by properties of ordinals). Then (i_0, x_0) is the least element of S. □

DEFINITION 287. $\sum a$ is the unique ordinal order-isomorphic to $\coprod a$.

EXERCISE 288. Prove that for finite ordinals it is just a sum of natural numbers.

This ordinal exists and is unique because our set is well-ordered.

REMARK 289. An infinite sum of ordinals is not customary defined.

The *structured sum* $\bigoplus a$ of a is an order isomorphism from lexicographically ordered set $\coprod a$ into $\sum a$.

There exists (for a given a) exactly one structured sum, by properties of well-ordered sets.

OBVIOUS 290. $\sum a = \operatorname{im}\bigoplus a$.

THEOREM 291. $(\bigoplus a)(n,x) = \sum_{i\in n}a_i + x$.

PROOF. We need to prove that it is an order isomorphism. Let's prove it is an injection that is $m > n \Rightarrow \sum_{i\in m}a_i + x > \sum_{i\in n}a_i + x$ and $y > x \Rightarrow \sum_{i\in n}a_i + y > \sum_{i\in n}a_i + x$.

Really, if $m > n$ then $\sum_{i\in m}a_i + x \geq \sum_{i\in n+1}a_i + x > \sum_{i\in n}a_i + x$. The second formula is true by properties of ordinals.

Let's prove that it is a surjection. Let $r\in\sum a$. There exist $n\in\operatorname{dom}a$ and $x\in a_n$ such that $r = (\bigoplus a)(n,x)$. Thus $r = (\bigoplus a)(n,0) + x = \sum_{i\in n}a_i + x$ because $(\bigoplus a)(n,0) = \sum_{i\in n}a_i$ since $(n,0)$ has $\sum_{i\in n}a_i$ predecessors. □