8°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a \wedge c \sqsubseteq b))$.

PROOF.

1°⇔2°⇔3°⇔4°⇔5°⇔6°. By the above theorem.

8°⇒4°. Let property 8° hold. Let $a \sqsubset b$. Then it exists element $c \sqsubseteq b$ such that $c \neq \bot$ and $c \sqcap a = \bot$. But $c \sqcap b \neq \bot$. So $\star a \neq \star b$.

2°⇒7°. Let property 2° hold. Let $a \not\sqsubseteq b$. Then $\star a \not\sqsubseteq \star b$ that is it there exists $c \in \star a$ such that $c \notin \star b$, in other words $c \sqcap a \neq \bot$ and $c \sqcap b = \bot$. Let $d = c \sqcap a$. Then $d \sqsubseteq a$ and $d \neq \bot$ and $d \sqcap b = \bot$. So disjunction property of Wallman holds.

7°⇒8°. Obvious.

8°⇒7°. Let $b \not\sqsubseteq a$. Then $a \sqcap b \sqsubset b$ that is $a' \sqsubset b$ where $a' = a \sqcap b$. Consequently $\exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a' \wedge c \sqsubseteq b)$. We have $c \sqcap a = c \sqcap b \sqcap a = c \sqcap a' = \bot$. So $c \sqsubseteq b$ and $c \sqcap a = \bot$. Thus Wallman's disjunction property holds.

□

PROPOSITION 223. Every boolean lattice is strongly separable.

PROOF. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a boolean lattice an $a \neq b$. Then $a \sqcap \bar{b} \neq \bot$ or $\bar{a} \sqcap b \neq \bot$ because otherwise $a \sqcap \bar{b} = \bot$ and $a \sqcup \bar{b} = \top$ and thus $a = b$. Without loss of generality assume $a \sqcap \bar{b} \neq \bot$. Then $a \sqcap c \neq \bot$ and $b \sqcap c = \bot$ for $c = a \sqcap \bar{b} \neq \bot$, that is our lattice is separable.

It is strongly separable by theorem 222. □

3.1.3. Atomically Separable Lattices.

PROPOSITION 224. "atoms" is a straight monotone map (for any meet-semilattice).

PROOF. Monotonicity is obvious. The rest follows from the formula

$$\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$$

(corollary 109). □

DEFINITION 225. I will call *atomically separable* such a poset that "atoms" is an injection.

PROPOSITION 226. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \operatorname{atoms} a \subset \operatorname{atoms} b)$ iff $\mathfrak{A}$ is atomically separable for a poset $\mathfrak{A}$.

PROOF.

⇐. Obvious.

⇒. Let $a \neq b$ for example $a \not\sqsubseteq b$. Then $a \sqcap b \sqsubset a$; $\operatorname{atoms} a \supset \operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ and thus $\operatorname{atoms} a \neq \operatorname{atoms} b$.

□

PROPOSITION 227. Any atomistic poset is atomically separable.

PROOF. We need to prove that $\operatorname{atoms} a = \operatorname{atoms} b \Rightarrow a = b$. But it is obvious because

$$a = \bigsqcup \operatorname{atoms} a \quad \text{and} \quad b = \bigsqcup \operatorname{atoms} b.$$

□

THEOREM 228. A complete lattice is atomistic iff it is atomically separable.